REPORT ON THE THEORY OF PROJECTIVE INVARIANTS: THE CHIEF CONTRIBUTIONS OF A DECADE.

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Introduction.

If we find it useful to distinguish short periods in the development of a science, the theory of invariants may easily enough be considered to have passed a milestone in 1887. In that year was published the second part of Gordan's Vorlesungen über Invariantentheorie. The plan of this work was dominated by the intent to expound and exemplify worthily the famous Gordan theorem on the finiteness of the form system of one or more binary forms. Gordan had announced and proven this theorem of fundamental importance in 1868.* and had since that time simplified his methods at least twice; and his was still in 1887, with one exception, the only current proof of the theorem. The two proposed by Jordan† and Sylvester‡ seem to have been not enough simpler to secure currency. The statement is, in briefest form, this: *For every binary form there is a finite system of covariants, in terms of which all other covariants, infinite in number, can be expressed rationally and integrally.* Without recalling here the details of the argument, we may characterize it as depending altogether upon the nature of the operations which generate covariants.

The one exception, just referred to, was a radically new method devised by Mertens, published in vol. 100 of the *Journal für reine und angewandte Mathematik.* By inductive process, assuming the theorem true for any given set of forms, he proves that it must still hold true when the order of one of the forms is increased by a unit. This method is deserving of attentive consideration, by virtue of its simplicity and power as shown in this first application, and even more on account of the strong probability that it might have been so extended as to prove the corresponding theo-

* *Crelle*, vol. 69.
† *Liouville's Journal*, 3d series, vol. 2 (1876), p. 122.
‡ *Proc. Lond. Math. Soc.*, vol. 27 (1878), p. 11–13.

rem for systems of ternary forms, and so in due course for forms in four, five, or any number of homogeneous variables.

The extension of Gordan's theorem to forms in more than two variables had not been achieved in the twenty years during which it had been eagerly awaited as an imminent possibility. For particular cases it was known to be true. Already in the first volume of the *Mathematische Annalen* (1869) Gordan had given out the complete system of ground forms concomitant to the ternary cubic: the system of a quadric in any number of variables was known, and the systems of two and of three ternary quadrics had been worked out by Gordan* and Ciamberlini† respectively. For a special ternary quartic:

$$f_4 = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1,$$

Gordan established a finite system of ground forms, 54 in number.‡ For further forms or sets of forms the systems had not been computed, nor was there any known proof that Gordan's method would terminate in a finite number of steps. Here lay the chief obstacle to further progress in the theory.

Two other problems of a general nature were plainly in need of study; the enumeration of covariants of given weight and of given order in three or more variables, and the construction of a systematic theory of syzygies. Why the former of these had not been completely solved it is hard to say. Sylvester's papers on the binary problem were practically concluded by Franklin's résumé in the third volume of the *American Journal of Mathematics* (1880), and these were obviously the model for subsequent investigation. And as to the latter, syzygies had been studied in connection with particular forms since the appearance of Cayley's second memoir on quantics (1856).§

It may fairly be claimed that the past decade has seen the solution of these three important problems. In many other points has the theory of invariants received valuable contributions, and in these three there is, of course, an immense amount of work needed in order to thoroughly possess the conquered territory. Postponing restrictions and qualifications and matters of secondary importance, let us consider briefly what has been done upon these three main questions.

*See Clebsch-Lindemann, Vorlesungen über Geometrie, I. (1876), p. 288; or Osgood in *Amer. Journ. of Math.*, vol. 14 (1892), pp. 262–273.

†*Battaglini's Giornale*, vol. 24 (1886), pp. 141–157.

‡*Math. Annalen*, vol. 17, pp. 217–233.

§Collected Works, vol. 2, p. 250–275.

§ 1. *Mertens' Demonstration and Hilbert's First Proof of Gordan's Theorem.*

It was evident that there was some advance movement under way when within thirty months (1886–88) there appeared two new brief and elegant demonstrations of Gordan's theorem for binary systems. The first depends upon symbolic expression of covariants, the second upon their expression in terms of the actual linear factors of the stem forms of the system.

According to Mertens, we can always adjoin a new linear form to any set of binary forms for which Gordan's theorem is known to be true, and it will still hold good for the enlarged set. And the set may be altered further by multiplying the linear form into any other form of the set, thus diminishing by one the number of forms, but raising by a unit the order of any one form. Evidently if unit changes of both these styles can be made without invalidating the theorem, we need only to know its truth for a single linear form—or, indeed, for a single form of order zero, a constant —before we can conclude its truth for a set of forms as many in number and as high in orders as we choose. And a linear form has no invariant or covariant except powers of itself: hence the theorem is universally true.

The first step, the adjunction of a linear form, is as follows: Gordan's development in series expresses any covariant in two sets of variables x_1, x_2; y_1, y_2, as a sum of a finite number of terms, each containing some power of the determinant $(y_2x_1 - y_1x_2)$ multiplied by a polar of some covariant which contains only the variables x_1, x_2. Now instead of y_1, y_2 insert the cogredient coefficient $-p_2, p_1$ of the adjoined linear form $(p_1x_1 + p_2x_2)$. Thus every covariant containing p_1, p_2 will be expressed in terms of $(p_1x_1 + p_2x_2)$ and a finite number of what we may still call polars derived from covariants not containing p_1, p_2. Accordingly these latter, by hypothesis finite in number, together with their polars in $(-p_2, p_1)$ and the form $(p_1x_1 + p_2x_2)$ itself, constitute the form system of the enlarged set of binary stem forms.*

The second step is almost equally simple, but introduces as auxiliary a Diophantine system of equations. If a form g is the product of a form f and a linear form p, then all covariants of a set including g are included among those of a set including f and p and all members of the first set except g. But the converse is not true: at most, only those

* Compare proof of the same result in Clebsch's *Binäre Formen*, §55.

covariants of $(\cdots, f, p)$ can be covariants of $(\cdots, g)$ which are of equal degree in the coefficients of f and of p. This condition yields a single equation, linear in the exponents of those fundamental covariants which appear in any term of a reducible covariant of $(\cdots, g)$. To state this more explicitly, call $F(\cdots, g)$ any covariant of the set indicated. In it replace coefficients of g by those of the product $f \cdot p$; $F(\cdots, g) \equiv F(\cdots, fp)$. The result is by hypothesis expressible in terms of a finite number of fundamental covariants of $(\cdots, f, p)$

$$F(\cdots, f \cdot p) = \Sigma c A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n} \cdot B_1^{\beta_1} B_2^{\beta_2} \cdots B_\nu^{\beta_\nu}.$$

Here every A_i is understood to contain the coefficients of f to a degree higher than those of p by some number, either zero or positive, which we may call a_i, the excess of A_i; let b_i denote similarly the defect of B_i. The equation of condition is then evidently

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = b_1\beta_1 + b_2\beta_2 + \cdots + b_\nu\beta_\nu.$$

This Diophantine equation has a finite number of linearly independent solutions in integers not negative, and to each corresponds one possible term in the above expansion; and to each reducible solution corresponds a term that can be factored into two or more of the non-reducible sort. But these non-reducible factors, though constituting a basis for covariants of the set $(\cdots, g)$, appear to involve still the irrational factor p of g. The means of removing this difficulty is not far to seek,* and so the theorem is proven for a set containing instead of f a form one degree higher, g.

Having analyzed at such length Merten's demonstration, we can state Hilbert's first proof with fewer words.† Instead of considering explicitly a single actual linear factor p of the stem form g, he considers all the factors

$$(x_1 - e_1x_2)(x_1 - e_2x_2) \cdots (x_1 - e_nx_2).$$

Every covariant is expressible rationally and integrally in these factors and in differences of the quantities $e_1, \cdots, e_n$ arising from the same stem form or from different stem forms. But the exponents of powers of such differences are parameters conditioned by a set of Diophantine equations, since the covariant is of equal degrees in the quanti-

* Precisely this rationalization is the most novel and ingenious feature of Mertens's proof.

† This demonstration is fully stated in Elliott's Algebra of Quantics, pp. 193-203. First published in *Math. Annalen*, vol. 33 (1888), p. 223.

ties e_1, e_2, ···, e_n, etc. To the independent integral solutions of these equations correspond a finite number of irrational integral functions of coefficients; and power products of these with all sets of exponents that can be formed within a certain finite range of integers are readily combined with their conjugates to yield a numerous but still finite system of covariants, a basis for the reduction of all others. This proof is simpler than that of Mertens in that, while each requires two logical steps, Hilbert's does not leave a subsequent step-by-step induction to be considered. Notwithstanding this, I venture to express a personal opinion that Merten's proof is the more powerful *as it stands*; for while there is very little difference in the ease of their application to covariants of a single stem form, for a greater number the method of Mertens divides the proof into its simplest possible elementary steps, all alike, while Hilbert prefers to consolidate it into a single argumentative process not repeated. Further, Mertens, employing factors which may be symbolic only, offers a possible opportunity for extension to forms in more than two variables, an opportunity not so readily discerned in Hilbert's use of *actual* factors.

§ 2. *Hilbert's General Proof of Gordan's Theorem for Forms in n Variables.*

It was an agreeable surprise to learn that the elaborate proofs of Gordan's theorem formerly current could be replaced by one occupying not more than four quarto pages. Gordan's series, required as a foundation in Merten's proof, was applicable to ternary forms; and it seemed entirely possible that by this attack might come the next considerable extension of the theorem. It is certain that no one was prepared for the announcement which came in December, 1888,* that the theorem could be established, by uniform method, for forms in any desired number of variables. No wonder that some learned heads shook in doubt over the sweeping generalizations of the enthusiastic young Dr. Hilbert from Königsberg. At length the most incredulous were obliged to concede that he had exemplified the maxim: Generalize your problem and solve it. Then it was apparent why the feat had not been accomplished before: investigators had been using tools much too fine for the work. Hilbert cast aside all needless limitations, and asked directly: If an infinite system of forms be given, containing a finite

**Göttinger Nachrichten*, 1888, pp. 450–457, and *Math Annalen*, vol. 36 (1890), pp. 521–529.

number of variables, under what conditions does a finite set of forms exist, a *basis*, in terms of which all the others are expressible as linear combinations with rational integral functions of the same quantities for coefficients? The answer was: Always. The argument need not be given here, for it is brief and simple, and is accessible either in Dr. Story's improved form in vol. 41 of the *Mathematische Annalen*,* in Meyer's, "Bericht über den gegenwärtigen Stand der Invarianten-Theorie,"† or in Weber's Algebra,‡ as well as in Hilbert's own memorable paper.§ We observe only that it proceeds inductively from the case of n quantities to $n + 1$; and that in the application to invariants it is a matter of indifference whether those quantities are variables, usually so called, or whether a part or all of them are coefficients of stem forms. The transformations to which the quantities are subject play no part in the argument until after the existence of a finite basis is established.

The application of the principal proposition is not of itself, however, sufficient for the requirements of Gordan's theorem. Every covariant F is reduced to the form

$$F \equiv A_1 F_1 + A_2 F_2 + \cdots + A_r F_r,$$

where $F_1, F_2, \cdots, F_r$ are covariants, but where the coefficients $A_1, A_2, \cdots, A_r$ are not known to be such. To transform the identical equation so as to substitute covariants for the A's without modifying the F's save by numerical factors, Hilbert devised or adapted a scheme of much intrinsic beauty, for which Dr. Story substitutes a most ingeniously contrived explicit differential operator.‖ The properties of this operator and its structural character, as necessary or arbitrary, are matters of importance that have not yet been discussed.

Thus after twenty-one years the question raised by Gordan's early success in binary forms is definitively settled, nor has there appeared as yet any proposal for a radically different proof of the theorem.

One important consequence of Hilbert's first theorem should be cited at this point. The generalization of a Diophantine system of equations will evidently arise by using, instead of constant coefficients and integral solutions,

* P. 471.

† *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 1 (1892), p. 145.

‡ Vol. 2 (1896), p. 165.

§ "Ueber die Theorie der algebraischen Formen," *Math. Annalen*, vol. 36, p. 475.

‖ l. c., p. 488.

homogeneous polynomials for coefficients, and requiring as solutions sets of homogeneous polynomials, rational in some prescribed domain. For such generalized sets of equations Hilbert shows that only a finite number of solutions are independent, in the sense that all others are compounded linearly from them, with coefficients rational in the same domain. The application of this corollary is primarily to the theory of syzygies.

§ 3. *Deruyts' Researches in Enumeration of Covariants of Given Characteristics.*

To state and to solve in the most general form the enumerative problem, the lowest case of which Cayley and Sylvester had at last brought to a conclusion, this was the aim of the series of studies which Deruyts consolidated into his now classic book.* Two features of the work first engage our interest. Discussing forms in n variables, he does not follow the older practice, due to Clebsch, of admitting $n-1$ different sets of variables, each set contragredient to one other (unless dual to itself) and cogredient to certain determinants formed from two, three, etc., rows of the variables of simplest type. Capelli's proposal is adopted instead, to employ $n-1$ sets of variables all cogredient. The older practice has intrenched itself in analytic geometry, and cannot be dislodged; but the alternative is undoubtedly better from the point of view of algebra. The second point is, that the whole argument is conducted by the aid of the Clebsch-Aronhold symbolic notation, and it is difficult to see how the end could be attained without this auxiliary.

Dr. Story has generalized Sylvester's semi-invariants by calling them differentiants, and distinguishing as many kinds of differentiants as there are pairs of variables in the set of n. Thus an xy-differentiant is invariant of the substitution

$$\begin{aligned} x &= x' + \lambda y' \\ y &= \quad\; y', \end{aligned}$$

the other variables remaining unchanged. Deruyts, on the other hand, uses the term semi-invariant, but enlarges the substitution with respect to which it is invariable. He defines semi-invariant as a function (properly qualified) which is not altered by a substitution in whose matrix all coefficients on one side of the principal diagonal are zero.

*"Essai d'une théorie générale des formes algébriques," par Jacques Deruyts; Bruxelles, F. Hayez, 1891, 8vo, pp. vi + 156.

Let this be the lower (left-hand) side; such a linear substitution is designated aptly by S_1. Semi-invariants are important for this reason, that each is the source of a *primary covariant*, which is uniquely determined by it; conversely every primary covariant has only one source. For example, the mixed concomitant or connex θ which occurs in the theory of the ternary cubic

$$\theta = (abu)^2 a_x b_x$$

is replaced in this theory by the primary covariant

$$\begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}^2 a_x b_x,$$

whose source is the semi-invariant

$$a_1 b_1 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2.$$

If we replace symbols of the cubic by those of its Hessian, we have a source which contains symbolic factors of all types that are possible in semi-invariants of ternary forms: the covariant becomes

$$(a_1 b_x c_3)^2 (d_1 e_x f_3)^2 \begin{vmatrix} a_x & d_x \\ a_y & d_y \end{vmatrix} \begin{vmatrix} b_x & e_x \\ b_y & e_y \end{vmatrix} c_x f_x$$

and its source is the semi-invariant

$$(a_1 b_x c_3)^2 (d_1 e_x f_3)^2 (a_1 d_x)(b_1 e_x) c_1 f_1.$$

Now the number of linearly independent covariants of given degrees and orders is the same as the number of primary covariants of like degrees and orders; and that again is equal to the number of different semi-invariants of corresponding type. From the covariants so determined, all others can be derived by the aid of polar processes and identical covariants such as $(x_1 y_2 z_3)$ or $(x_1 y_2 z_3 w_4)$, etc. The problem so reduced Deruyts has solved by means of partition numbers, so that the enumeration of *irreducible* covariants for given weights presupposes that for all lower weights and degrees. As an illustration of the computations used, consider the well known quadric covariants of second degree in the coefficients of each of two quadric forms in three variables

$$(abu)\ (ab_x\bar{3})a_x\bar{3}_x, \quad (a_1\bar{3}a)\ (a_x\bar{3}b)\,a_x b_x,$$
$$(aba)^2\bar{3}_x^2, \qquad (a_1\bar{3}a)^2 b_x^2,$$

which are connected by the single linear relation

$$(aba)^2\beta_x^2 - (a\beta a)^2 b_x^2$$
$$\equiv (aba)\,(ab\beta)a_x\beta_x - (a\beta a)\,(a\beta b)a_x b_x.$$

There should be found, as the number of covariants of these degrees and order, three. Deruyts' final formula gives

$$\begin{aligned}[422] &= \{422\}_\Delta \\ &= \{422\} + \{530\} + \{611\} \\ &\quad - \{431\} - \{512\} - \{620\} \\ &= 24 + 6 + 6 - 16 - 12 - 5 = 3.\end{aligned}$$

The expansion of a three rowed determinant Δ serves as a mnemonic for the formula. So completely is the problem resolved, that one can determine the required number, not only when the stem forms and concomitants contain $n-1$ sets each of cogredient variables, but even when more different sets are present, and when different groups of sets are subject to independent linear substitutions.

The next desideratum in this direction is a table showing the nature of the irreducible covariants in the system of the ternary quartic, a similar one for the quaternary cubic, and later for the simplest simultaneous systems and for stem forms that contain more than one set of variables. For the use of geometers, also, it would appear worth while still to consider whether the general problem might not be solved if Clebsch's $n-1$ sets of correlated variables were admitted in the stem forms, and whether the problem so stated can be made to depend in any way upon the results of the work of Deruyts.*

§4. *Hilbert's Theorem upon Syzygies of Higher Orders.*

Between the irreducible ground forms of a system arising from one or more stem forms there exist relations called syzygies. All terms of such a relation being collected in one member of the equation, the aggregate is termed a *syzygant*. A syzygant of the first kind is identically equal to zero, not when it is expressed in terms of the ground forms, but only when these are further reduced to terms of

* M. Deruyts kindly informs me (Dec. 10, 1898) that he has considered this question in a special memoir, and has found that for ternary forms there is a one-to-one correspondence between covariants in the two sorts of reduced systems, but that this is not so for stem forms in more than three variables. See his essay: "Sur la réduction des fonctions invariantes dans le système des variables géométriques," *Bull. de l'Acad. roy. de Belgique*, 3d series, vol. 24 (1892), pp. 558–571.

the coefficients of the stem forms. Syzygants of the first kind form a system similar to that of the covariants, out of which may be selected, as Hilbert proves, a reduced system of ground syzygants, in terms of which all others may be expressed as rational and integral functions. For the annihilation of this first kind of syzygant, to repeat what was just now said, the coefficients and variables of the stem forms constitute the requisite domain of rationality. But if the ground forms of the system of covariants be taken as constituting a second domain of rationality, in this new domain there will be annihilated certain linear functions of the syzygants of the first kind, and these are called syzygants of the second kind. Preserving, thenceforward, the same second domain of rationality, there rise successive kinds of syzygants one beyond another, each linear in the coefficients occurring in the kind next lower in rank. Recalling the theorem or corollary cited at the close of §2, we understand that each of these kinds of syzygants must constitute a finite system, in the sense that its ground forms are finite in number. For the first system there have been published two exhaustive methods of discovery, the first by Study in his concise and comprehensive "Methoden zur Theorie der ternären Formen,"* the second by Stroh,† in addition to that arising from the "typical representation" of binary forms. Upon kinds higher than the first there has been done practically no detail work. So much the more noteworthy is therefore the fundamental theorem disclosed by Hilbert (l. c., p. 492, Theorem III.) that *the number of kinds of syzygants is always finite*. If m denote for any given system the number of ground forms, then the successive kinds of syzygants are not more than $m + 1$ in number.

The proof, as in the case of the theorem discussed in §2, is entirely divorced from the processes peculiar to the theory of invariants, concerning itself only with rational integral functions as such, and yielding therefore as much to the theory of algebraic loci as to the knowledge of invariants. Although this proof is, as both Hilbert,‡ and Franz Meyer§ testify, "nicht mühelos," yet it is possible to convey briefly some idea of the scheme employed. Suppose arbitrary polynomials in m homogeneous variables to be denoted by

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*Leipzig, Teubner, 1889. See p. 97.

†"Ueber die symbolische Darstellung der Grundsyzygaten einer binären Form sechster Ordnung, u. s. w.," *Math. Annalen*, vol. 36 (1890), pp. 262–303.

‡l. c., p. 492.

§*Jahresbericht der deutschen Mathematiker-Vereingung*, vol. 1, p. 148.

$$\begin{Bmatrix} F_{11}, F_{12}, \cdots F_{1m_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ F_{n1}, F_{n2}, \cdots F_{nm_1} \end{Bmatrix},$$

and undetermined polynomials, rational in the same domain, to be denoted by $X_1, X_2, \cdots, X_{m_1}$ seek solutions to the set of equations

$$\begin{aligned} F_{11}X_1 + F_{12}X_2 + \cdots + F_{1m_1}X_{m_1} &= 0 \\ F_{21}X_1 + F_{22}X_2 + \cdots + F_{2m_1}X_{m_1} &= 0 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad\quad &\vdots \\ F_{n1}X_1 + F_{n2}X_2 \quad \cdots + F_{nm_1}X_{m_1} &= 0. \end{aligned}$$

(We pause to observe that if the F_{ik} are syzygants of the first kind, every solution of one such equation gives a syzygant of the second kind.) Suppose this set of equations fully solved, and a minimum sufficient set of fundamental solutions determined, two steps theoretically possible. Arrange in a rectangle these solutions, with values of X_1 in the first row, of X_2 in the second, etc. Denote this array by the symbols

$$\begin{Bmatrix} F'_{11} \cdots F'_{1m_2} \\ \vdots \quad\quad \vdots \\ F'_{m_1 1} \cdots F'_{m_1 m_2} \end{Bmatrix}, \text{ values of } \begin{Bmatrix} X_1 \\ \vdots \\ X_{m_1} \end{Bmatrix}.$$

Now seek relations linear in these horizontal rows of F's, just as before among the original F's. (Such relations will have for significant members the coefficients in syzygants of second kind, in the present application.) Continue this process, forming a second derived set, a third derived set, and so on until, if ever, the last set admit no solutions.

To see that this interruption will come, divide the solution of every set into two parts, such that the first part are immediately discoverable and help in reducing the order of the others in respect to a selected variable, say the last or m-th; while the second part depend for their determination upon a new set of similar equations, containing only the first $m-1$ variables. The device is so chosen that the second of these auxiliary sets is derived from the solutions of the first auxiliary set in the same way as each set in the principal series is derived from its preceding set in the same series. Argue now by induction: if the second series depending upon $m-1$ variables, is interrupted by default of solution after $m-2$ steps, then the principal series must be interrupted after $m-1$ steps. For complete assurance examine the form of the first part of solutions of the principal set, which will be after $m-1$ steps the *complete* array of solutions. It is

$$\begin{array}{lccccccc} & \varphi_{11} - x_n & \varphi_{12} & \varphi_{13} & \cdots & \varphi_{1\mu} & \varphi_{1\mu+1} & \cdots \\ & \varphi_{21} & \varphi_{22} - x_n & \varphi_{23} & \cdots & \varphi_{2\mu} & \varphi_{2,\mu+1} & \cdots \\ & \vdots & \vdots & \vdots & & \vdots & & \\ & \varphi_{\mu 1} & \varphi_{\mu 2} & \varphi_{\mu 3} & \cdots & \varphi_{\mu\mu} - x_n & \varphi_{\mu,\mu+1} & \cdots \\ \text{values of} & x_1, & x_2, & x_3, & \cdots & x_\mu, & x_{\mu+1} & \cdots \end{array}$$

Here the φ_{ik} denote functions of the first $n-1$ variables. If these be taken by columns for coefficients of the next derived set of equations, the determinant of the first μ not vanishing, obviously there can be no solutions.

The theorem is, therefore, true for any number of variables so soon as it is true for a single variable, *i. e.*, for equations in numerical constants. If we limit the field of rationality to the natural domain, this is in effect to reduce the inquiry to the question: whether linearly independent sets of solutions of a set of Diophantine equations are linearly independent! Hilbert chooses rather to establish a foundation proof for binary equations, introducing unnecessary complication, excusable on account of the elegance of his independent demonstration.

On this particular part of Hilbert's ample contributions to the theory of invariants I have dwelt at some length, though giving only the bare outline, and omitting entirely the critical points, first, because it has received less notice and excited less discursive activity than the more elementary theorems announced in the same essay; and because, in the second place, it serves admirably to illustrate the statement that it is time for the theory of invariants to attach itself firmly to the most modern developments of algebra. The sequel to this essay, a treatise on the production of complete fundamental systems of covariants.* has pushed the frontier in this direction a long distance ahead, establishing the main thesis: that a finite number of trials of perfectly definite kind will always lead to the knowledge of the complete system of ground forms when the stem forms are given. Kronecker's theory of entire algebraic functions proves itself indispensable and effective, and the argument leading up to the definition of canonical null forms is likely to become the standard concrete illustration of Kronecker's highly abstract theory.

§ 5. *Miscellaneous Topics.*

One other question of principal moment has been discussed, by Maurer, in vol. 107 of the *Journal für reine und*

* D. Hilbert: "Ueber die vollen Invariantensysteme." *Math. Annalen*, vol. 42 (1893), pp. 313-373.

angewandte Mathematik:* the division of stem forms and systems into classes according to the number of conditions satisfied by their coefficients, and equivalence of forms within classes. The subject seems to promise more interest when better developed.

Not less important for the growth of the science than original articles is the preparation of treatises and text books. Of these, at least two of high grade beside that of Deruyts, have come to my notice within this decade, those of Study and Elliott already cited above. Study's book introduces substantial improvements in notation, and gives precision to the notion of rationality, and is full of originality in every chapter. Elliott's is intended less for advanced students, but is admirable pedagogically. Others that I have seen announced are evidently elementary books for beginners.

Of value higher than text books, as every scholar understands, are exhaustive *résumés* and reference compends. For such a work, replete with description, discriminating and impartial in its estimates, students of invariants are indebted to Professor W. Franz Meyer, now of the University of Königsberg.† Undertaken at the instance of the Deutsche Mathematiker-Vereinigung, it has placed in a favorable light the utility of such coöperative organizations.

As showing the ample attention that is paid to this department of mathematics, it is of interest to note the number of titles, reviewed in the *Jahrbücher über die Fortschritte der Mathematik*, which can fairly be classified under theory of invariants. There were:

in 1889,	46	titles,
" 1890,	42	"
" 1891,	41	"
" 1892,	38	"
" 1893 and '94,	50	"
" 1895,	30	"
" 1896,	30	"

Of special papers in this field, since the important one of Story's referred to above, the most interesting one produced in this country is without doubt the recent essay by F. Morley in vol. 49 of the *Mathematische Annalen*, wherein he gives the long wished for geometric construction of the linear

*Ueber Invarianten-Theorie, pp. 89–116.

†" Bericht über den gegenwärtigen Stand der projektiven Invarianten-Theorie im letzten Vierteljahrhundert;" in the *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 1 (1892), pp. 79–292.

covariants of a binary quintic. This skilful synthesis removes from geometers the reproach which it is said Clebsch used in his lectures to cast upon them, in that none of them had yet been able to derive uniquely and symmetrically a sixth point from five given points on a straight line. The zeros of the quintic are denoted in Professor Morley's construction by five arbitrary points upon a conic.

§6. *Desiderata, and Remarks upon Courses of Instruction.**

Two things appear to me as proximate possibilities, and essentials to uniform advancement. Those familiar with Lie's group theory and interested in differential invariants will no doubt criticise this choice, which is perhaps in a narrower field.

(1) The working out of complete systems of syzygies of the first kind, second kind, and all higher kinds which occur among the covariants of binary forms of lowest orders. For the quintic and sextic the first kind are already tabulated by Stroh. This work will give tangible examples for the understanding and estimation of Hilbert's great theory.

(2) The revision of complete form systems already known, with the object of discovering subordinate systems among them. The most obvious point of attack, if we except the suggestive processes used in Clebsch and Gordan's classic treatise on the ternary cubic in vol. 6 of the *Mathematische Annalen*,† is offered by polars, symmetric in two sets of variables, derived from binary covariants of even order, and by ternary concomitants whose order and class are equal. By using these as transformers, systems of covariants can certainly be determined which are closed; and particular covariants ought to be looked for, which shall be automorphic under such transformations. The next step would be, by transformers which are analogous to these in all save that they raise the order of the operand, to produce infinite series of covariants and to discover their recurrent laws. These again might be expected to develop some subordinate closed systems, and others probably coextensive with the complete form system.

Finally, there is to be remembered the least explored and most fascinating portion of the field, equally promising to

* With regard to the following paragraph it should be explained that the Sectional Committee of Section A of the American Association for the Advancement of Science expressly requested of authors of reports the formulation of problems suitable for coöperative attack and of pedagogic theses inviting discussion. H. S. W.

† Pp. 436–512 (1873).

the analyst and the geometer, the realm of irrational covariants.

Upon the question of courses of instruction I wish to formulate two propositions:

(1) A first or elementary course in invariant theory ought never to be restricted to binary forms.

(2) Preliminary to or concurrent with an advanced course, there should be given courses in the theory of substitution groups or abstract groups, and in the algebra of modular systems and of entire functions.

EVANSTON, ILL.,
August, 1898.

REYE'S GEOMETRIE DER LAGE.

Lectures on the Geometry of Position. By THEODOR REYE, Professor of Mathematics in the University of Strassburg. Translated and edited by THOMAS F. HOLGATE, M.A., PH.D., Professor of Applied Mathematics in Northwestern University. Part I. New York, The Macmillan Company, 1898. 8vo, xix + 248 pp.

THE true geometry of position has hardly been accessible in English up to the present time. Townsend's Modern Geometry and Lachlan's Modern Pure Geometry are vitiated by the use of the circle, they are essentially metric; Cremona's Projective Geometry, in Leudesdorf's translation, is curiously uninteresting and unattractive, and does not seem to take the student sufficiently into the heart of the subject. Russell's Pure Geometry follows the French treatment of cross ratio, which is based on apparently metric relations, though it is shown that these relations are such that the metric quality is eliminated. Thus while it is a thoroughly useful book, it only gradually frees the student from the limitations of Euclidean geometry, instead of enabling him to walk at liberty from the first. It is possibly one of the easiest books to read on the subject; grafting the new ideas on to those already established, it expresses the unknown in terms of the known, whereas the more correct and satisfactory treatment, building up geometry *ab initio*, is apt to strike a student at first as an elaborate and artificial expression of the known in terms of the unknown. But while the grafting of projective geometry on

to metric may at first commend the subject, as the student advances and finds that the metric aspect is gradually disappearing, he cannot help feeling the essentially illogical nature of the method, namely, that the treatment is based on something that is afterwards shown to be irrelevant. It is a little too much like the game with which our elders were wont to puzzle our childhood, in which an elaborate series of arithmetical operations, performed on a secretly chosen number, brought out a result that had no relation to the initial number.

As regards the suitability of any particular presentation of a subject for teaching purposes, an appearance of strangeness in the preliminaries is a very small disadvantage, not to be compared with the feeling of uncertainty caused by a later discovery of a serious fault in the logic. Thus while the French treatment of modern geometry is natural and logical when the subject is approached analytically, it is to the standard German text books that we must turn at present for the pure geometrical treatment. Von Staudt and Reye are the names that at once suggest themselves; neither of these has been hitherto accessible in English, though Henrici's article on geometry in the Encyclopædia Britannica closely follows Reye's Geometrie der Lage (Leipzig, 1866–1892); I do not know of any English account of von Staudt's writings, the Geometrie der Lage (Nuremberg, 1847) and the Beiträge zur Geometrie der Lage (Nuremberg, 1856–1860). Probably Reye has thirty readers to von Staudt's one; his book is to be obtained without trouble, while von Staudt's is hardly in circulation; it is clearly printed in well chosen type pleasant to the eye, and great attention has been paid to the spacing and paragraphing, in marked contrast to von Staudt; moreover, Reye's sympathetic style is such as to commend the subject. Hence it is not surprising that Professor Holgate, wishing to make some such presentation of modern pure geometry available for students that prefer to face their mathematical and linguistic difficulties separately, should take up Reye's Geometrie der Lage as a matter of course. This work has undoubtedly great merits as a text book; but the lecturer must be prepared to deal with some flagrant logical lapses. Intellectual sincerity forbids that these be passed by without notice; yet I have seen that a frank recognition of their existence shakes a student's faith in the author, and diminishes his interest in the subject. I was interested to find last year, in discussing the matter with a distinguished Italian geometer, that he has felt precisely this difficulty, so strongly that he

has now adopted von Staudt in place of Reye with good results. As he remarked, von Staudt may be difficult, and may throw some hard work on the lecturer, but in arrangement and thought he is quite as interesting as Reye, and he is always absolutely logical.

It is with much diffidence and some feeling of ingratitude that I venture to criticise Reye's Geometrie der Lage, a book from which I have derived so much pleasure and profit. The objections urged relate only to the more elementary part, which can fairly be compared with von Staudt; without wishing in any way to disparage Reye, it does seem a cause for regret that von Staudt should be virtually ignored in college work; and the fact that there is now a readable English version of Reye will inevitably push von Staudt still more to one side.

Von Staudt seems to approach the subject in this manner. We have before us the visible universe; this, in so far as it is visible, we will treat geometrically. But we cannot handle any part of it, and so we cannot measure anything. The objects we see, or might see (for this is really what he means by *denkbar* at the beginning of the geometry) are bounded portions of space, hence solids; from these we obtain bounding surfaces, then bounding lines, and finally bounding points; and each of these can move in the preceding one of the series, so generating it. Straight lines and planes are defined as lines and surfaces having a particular property. The objects of our researches are therefore solids, surfaces, lines, and points; and all visible relations of these are matters for investigation. Any one of these objects is considered as having an individual unalterable existence, and as having position, which can be changed. The fact that there are lines in a plane that do not intersect is derived from observation; these lines are called parallel. Parallelism is shown to be equivalent to equality of direction, direction itself being accepted as a given idea, with no definition; it is shown that the direction of a line imposes on the line conditions that for all purposes are such as are imposed by a point. It is then shown that adjoining to the points already given by our view of the universe some others, namely, ideal points, the fact that lines and planes have a given direction and disposition (aspect) can be expressed by saying that they pass through certain of these ideal points. It is shown that the ideal points must be spoken of as all lying in a certain plane, the ideal plane or plane at infinity; this gives us in every plane one ideal line, in every line one ideal point;

and it is proved that for all purposes these ideal elements are to be treated in the same manner as those actually given us by the visible universe. Similarly imaginary, or feigned, elements are introduced and defined; and the proper form of reference to these feigned elements being investigated for every combination in which they can occur, it is found that they must be referred to in precisely the same manner as real elements. After thus justifying their treatment as actual geometrical elements, von Staudt explicitly extends the domain of the visible universe by adjoining to it these formal elements, both ideal and feigned. Thus his introduction of infinitely distant elements is absolutely logical; he adopts a particular phraseology to express a visible phenomenon, and proves that the verbal consequences of this phraseology lead us to correct conclusions. This is a perfectly logical method.

But in Reye's geometry, where the elements (point, straight line, and plane) are given in an arbitrary manner, without any statement that they visibly exist and without any definition, logic demands a different treatment. Nevertheless, in Lecture II, Reye speaks as though his geometry were derived from vision; he says that a case (of the intersection of two lines) is apparently exceptional because the point is lost to view, and then he introduces the line at infinity to account for this, and to show that the case is not really exceptional. This is altogether illogical. If the straight line at infinity is to be used at all, it ought to be postulated, absolutely and arbitrarily, in the same manner as the three elements. The same difficulty presents itself in connection with the idea of perpendicularity. The first use of this for purposes of argument* is on p. 115† (R. 108), where the nature of a right angle and the properties of a circle are assumed without question; we are not even told what definition of a circle is used, still less is any justification offered. The truth is, Professor Reye in his Geometrie der Lage has not the courage of his convictions. If his geometry is to be a purely logical cold-blooded system, with only intellectual justification or interest, built up from the three elements which alone he postulates. parallelism and perpendicularity have no place in it, and their surreptitious introduction discredits the whole system. To justify the recognition of these conceptions, something more must be

* The introduction of the circle on p. 49 (R. 49) is admissible, for it occurs in an avowedly metric supplement.

† The references to pages are here given in general for both the translation and the original, the latter being indicated as R.

given, arbitrarily and absolutely; and this something ought apparently to be a fixed quadric. This might perhaps be given in the first instance only for the plane, and in the degenerate form, as a pair of imaginary points, these determining the line at infinity in that plane; but the treatment would be certainly more satisfactory and possibly more simple if the absolute were introduced at once in its most general form. Whether a purely logical and intellectual geometry, thus constructed, would ever appear to an average student as in any way applicable to anything, is altogether another question and quite irrelevant; the system would at any rate be precisely what it claimed to be—synthetic geometry, built up honestly from the given elements without any extraneous help.

Objection on the score of logic may be taken also to the handling of the principle of reciprocity. This principle is really involved in the stated properties of the undefined elements, point, line, and plane; for it is mentioned in detail that the relation of point or line to point, line, and plane is precisely the same as that of plane or line to plane, line, and point, and thus if figures are built up from points, lines, and planes, other figures can be built up by the same law from planes, lines, and points. And yet on p. 30 (R. 29), Reye shrinks from assuming the general validity of the principle without demonstration, and says that it will be proved later; and on p. 102 (R. 97), in dealing with the polar relation in a plane, he says that this proves the principle of reciprocity for primitive forms of two dimensions.

But whatever objections may be urged against some things in the book, its general charm, to which is partly due its acceptation as the standard text book, is undeniable, and it was inevitable that sooner or later it should be brought out in English. Professor Holgate has translated Part I from the third edition (1886); doubtless Parts II and III, which carry the subject far beyond its elements, will follow. The division into lectures is retained, with the additional advantage over the German original of the numbering of the articles. Here and there it is difficult to determine on what principle this numbering has proceeded; §§ 41 and 42 on pp. 28, 29, should not have been separated; § 82 on p. 55 is inserted in the middle of one of the lines of the original. The translation is distinctly readable, and is in general very carefully executed. Some Teutonisms are to be found, for example, bisection point (p. 107), and intersection point, involution curve, and involution cone (p. 152). Such constructions as "the infinitely distant

point lies out in both directions upon the line" (p. 18), "we can associate in certain figures, to each vertex, * * * a corresponding element of another" (p. 34), "rays or planes * * * may be sectioned by a straight line" (p. 40), "relate the sheaves * * * perspectively to the range of points" (p. 187), are hardly admissible. *Weitläufig*, as used on p. 37 (R. 36) is not exactly profuse, but rather diffuse or prolix; on pp. 2, 33, 35, and many others, *wird* should be will, not would, in accordance with English usage. These do not affect the meaning; but there are two passages where the precise sense of the original has been slurred in the translation. On p. 12 (R. 12, 13) "we have *included* angles as part of a sheaf" is given as the equivalent of "wir * * * den Winkel als Theil eines Büschels *definirt* haben;" and on p. 130 (R. 121), where Reye evidently intends to define the class of a regulus, this being the first time he has occasion to make use of it, he says: "Die Regelfläche ist von der zweiten Classe," which is translated "The surface is *thus* of the second class," as if the class of a surface had already been defined.

These are comparatively slight defects, such slips as are almost sure to occur in a translation; they are perhaps small matters to single out for notice from 248 pages, but when a book is translated, the details of the translation invite criticism, however much we may appreciate the unselfishness of the translator in expending so much time and trouble on so altruistic a task. There remains however a more important matter to discuss, that of terminology. The whole question of nomenclature in mathematics is a difficult one; for as primarily disconnected subjects are found to overlap, the characteristic terms may turn out to be inconsistent. Thus it is impossible to say that no one individual shall be allowed to change an established term, for a change may be imperative. But certainly no one has a right to change a term without ample justification, and it is difficult to see what reasons Professor Holgate can adduce strong enough to justify him in discarding the well established and universally understood *pencil* (Büschel) in favor of sheaf, and replacing *sheaf* (Bündel) itself by bundle. Doubtless these terms are in themselves just as good; but the others are already and have been for many years in possession; they are perfectly distinctive and euphonious, hence the change is unnecessary, and therefore to be deprecated as likely to cause confusion. Another danger to be guarded against is the unnecessary naturalization of foreign words; sheet is as good as nappe (p. 93); skew is used for *schief* (p. 53), why not also for

windschief (p. 26) instead of gauche? the two words have practically the same signification. The word *Schein*, so significant in the German, is apparently untranslatable; Professor Holgate adopts projector as the equivalent; this answers sufficiently well in Reye, but misses the point in von Staudt's use of the word (Geometrie der Lage, § 3, p. 12), where the reference is explicitly to the *visual* foundation of the geometry. The translator of von Staudt will be hard put to it to render the term adequately.

The book is clearly and accurately printed, but is spoilt for pleasant handling by its most unusual weight.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR,
September, 1898.

BURKHARDT'S THEORY OF FUNCTIONS.

Funktionentheoretische Vorlesungen. Von HEINRICH BURKHARDT. Erster Teil: *Einführung in die Theorie der analytischen Functionen einer complexen Veränderlichen.* Leipzig, Veit & Co., 1897. 8vo, xii + 213 pp.

THE object of the author in writing the little volume before us has been to furnish an introduction to the theory of functions which is not confined to the presentation of the methods of any one school (Cauchy, Weierstrass, Riemann) but blends these methods as far as possible into an organic whole. The author has been very successful in making his book an introduction not merely to those parts of the theory which have long been classical (algebraic, elliptic, and Abelian functions) but also to the many other important developments of the last thirty years.* The mathematical public may well congratulate itself that a mathematician so thoroughly familiar with all sides of the subject as is Professor Burkhardt has undertaken the task of writing an elementary work along these lines.

We will briefly indicate the subjects treated.

Chapter I is an excellent presentation of the elementary theory of complex numbers and their geometric representation, in which the author has wisely restricted himself to the ordinary complex numbers $a + bi$. It is interesting to

* We note, for instance, the introduction of the terms *automorphic functions*, *fundamental region*, and the proof and applications of the law of symmetry.

notice that here as in Weber's Algebra the complex number is regarded as a "couple" of real numbers. Thus two of the most recent and most excellent German text-books have returned to the point of view familiar in England more than fifty years ago.

Chapter II on rational functions is of an elementary algebraic and geometric character. The linear function is studied in some detail here, the elliptic, hyperbolic, and parabolic cases being, for instance, treated, although unfortunately no names are given to them.*

Chapter III is devoted to a statement of some definitions and theorems (no attempt being made to prove these latter) concerning real variables and their functions. We fear that most young readers will find many parts of this chapter so difficult as to be of relatively slight service to them. For other reasons also, to which we shall come back later, this chapter seems less happily conceived than the rest of the book.†

Chapter IV deals with single valued analytic functions of a complex variable. Besides the familiar matters on integration and development in power series found in all text-books, will be found Weierstrass's theorem that a uniformly convergent series of analytic functions is analytic and can be differentiated term by term. The admirable proof here given by means of Cauchy's integral formula is I believe

* §12 which is contained in this chapter is destined, I am afraid, to do very great harm to young readers and this in spite of the fact that there is nothing incorrect in it. This section is devoted to the definition and discussion of the symbol ∞. This symbol is defined as a new number by the relation $a/o = \infty$. This is of course perfectly allowable, but it is in the highest degree undesirable because in the first place "∞" as thus defined is of no use, nor can it even be regarded as fulfilling the purpose for which it was introduced, viz., to make the operations of addition, subtraction, multiplication, and division always possible. Before "∞" was introduced there was one exception, division by zero was impossible; afterwards there are five exceptions, the operations $0/0$, ∞/∞, $0 \cdot \infty$, $\infty \pm \infty$ being impossible. What is the gain even from the most abstract point of view? In the second place the ∞ of analysis is a very different thing from this useless "∞" and there is the greatest danger of confusing the two. The ∞ of analysis is a *variable* which increases indefinitely, not a new kind of *constant*. The student is only too prone to think of it as a very large constant and this section will do much to confirm him in this wrong idea.

† A detailed discussion of the contents of this chapter would carry us too far. Two oversights of a somewhat serious nature must however be mentioned. In §25 theorem VI is not correctly stated (Cf. for instance Tannery et Molk: Fonctions elliptiques, vol. 1, §16, p. 23), and in §26 theorem XV is not correct after the definition of a continuous curve contained in theorem X (Cf. the paper by Peano, *Math. Annalen*, vol. 36, p. 157, or for a simpler presentation the paper by Hilbert, *Math. Annalen*, vol. 38, p. 459).

given in only one other text-book.* The chapter closes with a simple form of Mittag-Leffler's theorem and applications to simply periodic functions.

Chapter V is on multiple valued analytic functions. The angle of a complex quantity is first considered as an infinitely multiple valued function and its Riemann's surface is constructed. We believe the author to be quite right in regarding this as the best introduction to the subject of Riemann's surfaces. The function $\log z$ defined as a definite integral is then discussed, and then come the functions $\sqrt{z}$, $\sqrt[n]{z}$ and a few other simple irrational functions.† The chapter closes with a special case of Weierstrass's theorem concerning the expansion of a function in an infinite product.

Chapter VI, entitled "The general theory of functions," introduces the ideas of analytic extension and natural boundaries and the law of symmetry with applications to the conformal representation of rectilinear and some circular triangles on a half plane.

An alphabetical index closes the book.

It will be seen that the author barely touches on the subject of algebraic functions and their integrals which usually play such an important part in books on the theory of functions. Herein we think he has been wise, for this subject is after all a very special one. Apart from this, however, it will be found that in spite of its small size the volume before us touches on more sides of the subject than most other treatises on the theory of functions.‡ From the

* Demartres, Cours d'Analyse, vol. 2, p. 74.

† On p. 167 the author says: "An algebraic function is not necessarily in itself simpler than a transcendental function," and hereby states an important fact only too often forgotten. I am however unable to agree with the author that the introduction he gives to the function $\sqrt{z}$ by using $\log z$ as an auxiliary function is a desirable one, any more than I should think a teacher of mechanical drawing right who taught, as the first method of drawing a straight line, the use of some linkage, on the ground that the manufacture of a straight ruler involves still more complicated principles

‡ There are, however, some unfortunate omissions. The side of the subject most closely connected with mathematical physics is barely touched upon (pp. 90 and 95); the fundamental theorem that it is always possible to find a function which in a given region is harmonic and has arbitrarily given values on the boundary not being even mentioned. Riemann's theorem that every simply connected region can be represented conformally on a circle ought surely to have been at least stated Moreover, the concrete examples of conformal representation are all too simple to bring out some of the important principles of the subject. The direct discussion of the integral at the end of §72, for instance, would have largely supplied this omission (Cf. Picard, Traité d'Analyse, vol. 2,

point of view of rigor also the volume has great merits. In this respect, however, it seems to the reviewer that a mistake is made here as in almost all other text books on this subject. It is almost universally considered that in a work on the theory of functions absolute rigor should be demanded even though the reader may have been accustomed up to this point to the utmost laxity. This I believe to be fundamentally wrong. The teacher (or the writer of text books) should insist at every stage of the pupil's development on as great rigor as the pupil is then able to appreciate. If this plan were followed elementary text books on the calculus would not abound with statements contrary to common sense (such for instance as that the differential of a function is the limit of its increment), nor, on the other hand, would writers on the theory of functions try to attain absolute rigor at all points. In other words the critical sense of the student should be gradually developed and even if this is properly done it cannot be expected that it will be fully developed when he begins the study of the theory of functions.

If what has just been said is correct, it follows that the attempt made in Chapter III to secure absolute rigor is unwise. Apart from this however it seems unfortunate to concentrate in this very brief form almost all the work contained in the book on the subject of exact analysis, as the reader can hardly fail to get the erroneous impression that when once the results of this chapter have been accepted it is not necessary to be on the lookout for any further difficulties of the same nature, especially as the author has usually arranged his work very skilfully so as to avoid such difficulties. Occasionally however this cannot be done and once or twice the author has overlooked essential points as in Theorem VIII, p. 92, which says that if in a region S of the z-plane $w = f(z)$ is a single valued analytic function which does not take on the same value twice and whose derivative does not vanish in S, then the values of w fill a region (Bereich) within which z is an analytic function of w. Now it is obvious that the region S corresponds in a one to one manner to a set of points (Punktmenge) in the w-plane, but how do we know that this set of points is a "Bereich" in the technical sense in which the word is here used? How do we even know that it is a two dimensional continuum, for that is the essential point? I know of only two methods

chap. X., §§ 9, 11). Another subject which might well have been included is the determination of functions satisfying simple functional relations.

which the author would have regarded as available by which this problem can be attacked: *first*, the method suggested by Briot and Bouquet (théorie des fonctions elliptiques §130), or the modification of this method suggested by C. Neumann (Abelsche Integrale, Chap. VI., §2); and *second* the theory of implicit functions of two real variables involving the use of Jacobians.* It is to be regretted that the author did not include a brief and elementary account of this last mentioned method, which has so many other important applications, rather than some of the more difficult parts of Chapter III.

After all has been said however the volume before us remains an excellent treatment of the subject; good as an introduction, in so far as it does not prove too difficult; excellent for the mature student who already knows something of the subject; and invaluable to the teacher.

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DARBOUX'S ORTHOGONAL SYSTEMS.

Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes. Par GASTON DARBOUX, Membre de l'Institut, Doyen de la Faculté des Sciences et Professeur de Géométrie Supérieure à l'Université de Paris. Tome I, Paris, Gauthier-Villars et fils, 1898. 8vo, i+338 pp.

THE present volume is the fifth which Professor Darboux has prepared during the last decade for the Course in Geometry of the Faculty of Sciences of the University of Paris. This new work is to be devoted to the exposition of a theory which has its origin in the writings of Lamé and which has been the subject of a large number of researches in recent years. It is a direct sequel to the author's admirable treatise on the theory of surfaces in which he presented incidentally various properties of orthogonal systems and curvilinear coördinates, but reserved the organic and systematic development of these theories for a separate treatise of which the above is the first volume. The work glistens with originality both in material and in modes of presentation; the exposition exhibits the elegance and clearness characteristic of the author's writings, and the volume, as

* Cf. Jordan, Cours d'Analyse, vol. 1, pp. 80-89.

a whole, reveals abundant additional proof of how worthily the mantle of Chasles is worn by Darboux.

Book I. The Equation of the Third Order.

1. The families of Lamé and Dupin's theorem with its reciprocal occupy the first chapter. The parameters α, β, γ of three orthogonal families, considered as functions of the rectilinear coördinates x, y, z of a point in space, satisfy a certain system of partial differential equations; a general theorem of Cauchy shows that the general solution of this system depends on three arbitrary functions of two variables. As an application of this theorem Darboux shows that we can determine a triply orthogonal system by the condition that the three families of surfaces which compose it intercept on a given surface given curves subject to the condition that the curves shall not cut at right angles. The elimination of one of the parameters leads immediately to the theorem of Dupin; surfaces composing a triply orthogonal system mutually intersect along their lines of curvature. The reciprocal of this theorem, namely the theorem: If we have two families of surfaces cutting orthogonally along lines which are lines of curvature for the surfaces of one of the two families, there exists a third family completing the orthogonal system, permits of proving that the parameter of every family forming part of a triply orthogonal system ought to satisfy a partial differential equation of the third order; this differential equation is at the same time necessary and sufficient, *i. e.*, every solution of it leads to a triple system. Every family of surfaces which can constitute a part of a triple system Darboux calls a *family of Lamé*. The existence of the defining differential equation of the third order was first recognized by Darboux* in 1866; its developed form was given a few years later by Cayley.† In order to facilitate the formation of this differential equation Darboux introduces various properties of the differential operator

$$\delta_u v = \delta_v u = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}. \qquad (1)$$

* Darboux, "Sur les surfaces orthogonales," *Annales de l'École Normale*, 1st series, vol. 3 (1866), pp. 97–141.

† Cayley, "Sur la condition pour qu'une famille de surfaces donnée puisse faire partie d'un système triple orthogonal," *Comptes rendus*, vol. 75 (1872), pp. 116, 177, 246, 324, 381, 1800; Collected Mathematical Papers, vol. 8.

He then establishes by an entirely new process that the whole difficulty of the problem of orthogonal systems is referred to the integration of the equation of the third order, which Darboux writes in the form of a very simple determinant of the sixth order. After verifying the results obtained previously by Bouquet* relative to families represented by an equation of the form $u = \varphi(x) + \psi(y) + \chi(z)$ and by V. Puiseux† relative to a particular system of axes, Darboux brings the differential equation under a new and irrational form by introducing the derivatives of the function H defined by the relation

$$\frac{1}{H} = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}. \qquad (2)$$

The chapter concludes with an application of the theory of characteristics of partial differential equations and of systems of such equations, for the case of three independent variables, to the problem in hand. The results furnish the theorem that we can always determine a triple system by the condition that the surfaces (A), which compose one of its families, cut a surface (Σ) along given curves (C) and have, along these curves, a contact of the second order with certain surfaces (S) which contain these curves, unless the curves (C) are lines of curvature of the surfaces (S), or the surfaces (S) are orthogonal to the surface (Σ); in these exceptional cases the problem becomes either impossible or indeterminate.

2. The second chapter studies triple systems containing a family of planes or a family of spheres. Every family of planes or of spheres is capable of forming a part of a triply orthogonal system. Those triple systems which contain a family of planes are obtained by drawing two families of rectangular curves in a plane and rolling this plane on any developable surface. Darboux shows also that all consideration of rolling may be avoided and these systems constructed without integration. After giving Lamé's definition of curvilinear coördinates and deriving the form of the lineal element of space for the preceding system, the author establishes equations capable of defining the system and containing no signs of quadrature. If the family of planes is given *a priori*, the determination of the orthogonal

* Bouquet, "Note sur les surfaces orthogonales," *Liouville's Journal*, 1st series, vol. 11 (1846), p. 446.

† V. Puiseux, "Note sur les systèmes de surfaces orthogonales," *Liouville's Journal*, 2d series, vol. 8 (1863), p. 335.

trajectories, and consequently that of the corresponding triple systems, depends on three quadratures. Those families of spheres, for which the problem of orthogonal trajectories leads to the same differential equations, Darboux calls similar systems of spheres. The families similar to a given family of spheres can be constructed without integration. Among the families thus determined there exists an infinite number composed of spheres passing through a fixed point; a simple quadrature constructs these particular families if an orthogonal trajectory of the original family is known. By virtue of the preceding theorems we can pass, without integration, from triple systems containing a family of planes to those containing a family of spheres. The determination of the orthogonal trajectories of a given family of spheres is referred to the integration of two equations of Riccati, of which each trajectory furnishes a particular solution. A beautiful geometrical construction of triple systems containing a family of spheres is followed by Darboux's extension of the preceding results to space of n dimensions. The extension is an immediate consequence of Darboux's theorem that by giving a suitable form to the functions $a_1, a_2, \cdots, a_n$ and r, we can assign the general integrals of the system

$$\frac{dx_1}{x_1 - a_1} = \frac{dx_2}{x_2 - a_2} = \cdots = \frac{dx_n}{x_n - a_n}, \tag{3}$$

in a real form and without the sign of quadrature, $a_1, \cdots, a_n, r$ being functions of a parameter t connected with $x_1, \cdots, x_n$ by the relation

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = r^2. \tag{4}$$

The above system (3) of differential equations enters into various investigations in geometry. Notably for $n = 4$ in the work of Serret* on surfaces having spherical lines of curvature and in a note of Bonnet's,† in which he shows that the functions $a_1, \cdots, a_n, r$ can be brought to such a form that the integration can be effected by simple quadratures. Darboux proves that the same result can be reached without the sign of integration and without introducing the imaginaries made use of in Bonnet's investigations.

3. The partial differential equation of the third order

* Serret, J. A., "Sur les surfaces dont les lignes de l'une des courbures sont sphériques," *Comptes rendus*, vol. 42 (1856), p. 109.

† Bonnet, O., "Note sur l'intégration d'une certaine classe d'équations différentielles simultanées," *Comptes rendus*, vol. 53 (1861), p. 971

which determines the families of Lamé admits of particular solutions defined by the equation of the first order

$$H = \varphi_0(u)(x^2+y^2+z^2) + \varphi_1(u)x + \varphi_2(u)y + \varphi_3(u)z + \varphi_4(u). \qquad (5)$$

The third chapter is devoted to the study of these solutions. The equation (5) includes as particular cases the equation which characterizes parallel surfaces and also that which characterizes the surfaces derived by inversion from a family of parallel surfaces. The orthogonal trajectories in the first case are straight lines, in the second, circles passing through a fixed point. In order to lighten the discussion of the general case Darboux begins by studying the particular problem in which the mutual ratios of the five functions $\varphi_i(u)$ reduce to constants. The corresponding families of Lamé are then defined by the following construction: We construct the circles normal to any surface (Σ) and to a fixed sphere (S); all these circles are normal to the surfaces (Σ') which compose the family sought; we construct each surface (Σ') point by point by determining the point upon each circle where the circle is normal to (Σ), the two points where it is normal to (S), and by constructing the fourth point which forms a constant anharmonic ratio with the preceding points always taken in the same order. The two other families of surfaces which complete the triple system are evidently formed of surfaces one of whose systems of lines of curvature is made up of circles. If this construction be applied to a cyclide of Dupin, a triple system is obtained composed exclusively of cyclides. After studying the special case, remarked by W. Roberts,* where the cyclides are of the third degree, Darboux shows that the preceding general construction gives a contact transformation characterized by the conservation of lines of curvature. Lie has determined the forms of all contact transformations possessing this property; analytically they are equivalent to an orthogonal linear substitution performed on the six coördinates of a sphere; Darboux presents an elegant geometrical résumé of the results of Lie's researches. Returning to the most general case of the equation (5) Darboux finds that the general integral can be given without the sign of quadrature. In order to interpret the solution geometrically the author gives certain fundamental properties of the function

*Roberts, W., "Application des coordonnées elliptiques à la recherche des surfaces orthogonales," *Crelle's Journal*, vol. 62 (1863), p. 57.

H, which, multiplied by du, represents the shortest distance between two infinitesimally adjacent surfaces in any family. It amounts to the same thing to give H for every point of a surface or to give in these points the osculating circles of the curves which are the orthogonal trajectories of the family. The families discussed in this chapter are characterized by the property that the osculating circles of the orthogonal trajectories at the points where they cut one of the surfaces, are orthogonal to the same sphere, which varies with the surface. Darboux closes the chapter with a paragraph indicating how these families may be generated by means of infinitesimal transformations. The reader will observe that a typographical error makes two consecutive sections on pages 66 and 67 bear the same number, "39."

4. The author proves that the partial differential equation of the third order can be obtained by expressing that the shortest distance between two infinitesimally adjacent surfaces of the family is a particular solution of a point equation relative to the conjugate system formed by the lines of curvature. This theorem, together with the geometrical interpretation given at the close of the preceding chapter, leads to the theorem of Ribaucour :—The osculating circles of the orthogonal trajectories at the points where they cut a determinate surface of the family form a cyclic system. After a beautiful geometrical demonstration of Ribaucour's theorem and its reciprocal, Darboux proposes to complete the study of cyclic systems made in his Theory of Surfaces by attacking the two new problems: 1° the determination of those families of Lamé for which the osculating planes of the orthogonal trajectories at the point where they cut one of the surfaces of the family intersect in a point; 2° the determination of the cyclic systems formed by circles whose planes envelop a developable surface; the solution of the latter problem is due to Ribaucour.* Maurice Lévy† has given a remarkable form to the equation of the third order by taking as independent variables the parameter u and two rectangular coördinates; Darboux derives this form from the general equation and applies it to finding those surfaces of invariable form which are capable of generating families of Lamé by movement. When the move-

* Ribaucour, A., "Mémoire sur la théorie générale des surfaces courbes," *Liouville's Journal*, 4th series, vol. 7 (1891), p. 264.

† Lévy, M., "Sur une réduction de l'équation à différences partielles du troisième ordre qui régit les familles de surfaces susceptibles de faire partie d'un système triple orthogonal," *Comptes rendus*, vol. 77 (1873), p. 1435.

ment of the surface is unique and determinate it is necessarily helicoidal. The case where the surface may produce a family of Lamé by several different movements has not been completely solved. From a result obtained by Lucien Lévy* Darboux deduces that the spheres and planes are the only surfaces which can generate a Lamé family by all possible displacements, a theorem established also by Goursat.† Bertrand‡ has shown by geometry that if the movements include all translations the surface is a sphere or a cylinder. Adam§ has shown analytically that this result obtains if the movements reduce to two different translations, and has thus given the most extended result yet known in the solution of the problem. The partial differential equation of the surface sought expresses that the straight line of the plane joining the centers of geodesic curvature of two lines of curvature belongs to a linear complex; we owe this elegant geometrical interpretation to Petot.‖ How faithfully Darboux has brought his work up to date is indicated by the fact that E. Cosserat's theorem,¶ appearing after the body of the work was in type, is included in an extensive footnote. The chapter concludes with the formation of the general differential equation for the case in which the family is determined by an implicit equation $\varphi(x, y, z, u) = 0$, and the development of the equation for subsequent application.

5. The fifth chapter deals with the families of Lamé formed by quadrics. By seeking the condition that quadrics having a unique center and the same principal planes form a Lamé family, Darboux finds a differential relation between the axes found by M. Lévy** in 1867. The author

* Lévy, L., "Note sur le déplacement d'une figure de forme invariable," *Bulletin des Sciences mathématiques*, 2d series, vol. 15 (1891), p. 76.

† Goursat, E., "Sur les systèmes orthogonaux," *Comptes rendus*, vol. 121 (1895), p. 883.

‡ Bertrand, J, "Note sur un théorème de Géométrie," *Comptes rendus*, vol. 121 (1895), p. 921.

§ Adam, P., "Sur les systèmes orthogonaux," *Comptes rendus*, vol. 121 (1895), p. 812.

‖ Petot, A., "Sur certains systèmes de coordonnées sphériques et sur les systèmes triples orthogonaux correspondants," *Comptes rendus*, vol. 112, p. 1426; "Sur les surfaces susceptibles d'engendrer, par un déplacement hélicoïdal, une famille de Lamé," *Comptes rendus*, vol. 118, p. 1409.

¶ E. Cosserat's theorem: The cyclide of Dupin can generate a family of Lamé by an infinite number of different movements; these movements result from the composition of two rotations about the straight lines D, Δ, through which the planes of the two series of lines of curvature of the surface pass. *Comptes rendus*, vol. 124 (June, 1897), p. 1426.

** Lévy, M., "Mémoire sur les coordonnées curvilignes orthogonales et en particulier sur celles qui comprennent une famille quelconque de surfaces du second degré," *Journal de l'École Polytechnique*, vol. 43 (1867), p. 157.

presents also Lévy's direct derivation and geometrical interpretation, namely, that the differential relation expresses that any one of the umbilical lines, *i. e.*, any one of the lines described by the umbilics, is normal to the surfaces composing the family. It follows from this that the families of quadrics sought are determined as soon as any plane line is given *a priori* which will serve as an umbilical line. The particular case where one of the umbilical lines reduces to a straight line has been found by G. Humbert;* then the eleven other umbilical lines are straight lines and the surfaces which compose the family are tangent to eight isotropic planes and form part of a point sheaf; we can determine the two other families which complete the triple system. After an interesting discussion of those cases in which the umbilical lines are circles, conics, etc., Darboux returns to the general problem and shows that it can be solved without using the umbilical line and by a direct method which can be attached to a general principle. The proposition relative to umbilical lines does not apply to families composed of surfaces of the second degree alone, but it can be shown that it is true for any surfaces forming a family of Lamé; the demonstration given by Darboux rests on the consideration of the form of the lines of curvature in the neighborhood of an umbilic. This general proposition leads to the result that, if quadrics having unequal axes form a family of Lamé, the principal planes of these surfaces necessarily coincide; hence the preceding investigations are not as particular in character as we should have supposed, but make known all families of Lamé formed of quadrics whose axes are unequal. Darboux demonstrates and applies to surfaces of the second degree a much more general theorem, namely the theorem that if a family of Lamé is composed of surfaces each of which possesses planes of symmetry, the planes of symmetry ought to coincide. He quotes a similar theorem† relative to anallagmatic surfaces, and then turns his attention to the concluding problem of the chapter. We know that the surfaces which are the loci of points such that the sum or the difference of their distances to two fixed surfaces (A), (B) is constant, form a doubly orthogonal system, *i. e.*, they are distributed into two dis-

* Humbert, G., "Sur les normales aux quadriques," *Comptes rendus*, vol. 3 (1890), p. 963.

† Darboux, "Mémoire sur la théorie des coordonnées curvilignes et des systèmes orthogonaux," *Annales de l'École Normale*, 2d series, vol. 7 (1878), pp. 136–138.

tinct families of orthogonal surfaces; the problem then is to find in what cases the system can be completed by adjoining to two different families a third family composed of surfaces cutting the preceding at right angles. Darboux demonstrates geometrically that if we neglect the solutions already studied, the third family ought to be composed of quadrics whose right line generators are normal to the surfaces (A), (B). The axes of these quadrics satisfy certain differential equations which appear in the theory of elliptic functions and which were first integrated by Halphen;* Darboux gives a new method of integrating these equations. In the case of surfaces without centers, the elliptic functions are replaced by logarithms, except in one case remarked by Serret in which the solution is algebraic: in this exceptional case the system is represented by the equations

$$yz = ux, \quad \begin{aligned} &\sqrt{y^2 + x^2} + \sqrt{z^2 + x^2} = v, \\ &\sqrt{y^2 + x^2} - \sqrt{z^2 + x^2} = w; \end{aligned} \tag{6}$$

Darboux characterizes this result as the most beautiful one of the theory and shows how it may also be obtained by geometry. The chapter closes with the proposal of the problem, still unsolved, of finding the lines of curvature of a surface which is the locus of points for which is constant the algebraic sum of their distances from two straight lines which are in a relation less particular than that of being rectangular and cutting; the case which this problem seeks to generalize is solved in the text.

6. The sixth chapter discusses the systems completely orthogonal in a space of n dimensions, and extends the methods of the preceding chapters to such systems. The first example of orthogonal systems in n variables is Jacobi's system of general elliptic coördinates. Darboux generalizes in a similar manner the system of orthogonal and confocal cyclides and uses for the case of n dimensions variables analogous to pentaspheric coördinates. The fundamental problem of the chapter is: Given n functions forming a system completely orthogonal; to eliminate all the functions except one, and to form the necessary and sufficient partial differential equations which the latter function ought to satisfy. Darboux's method of attack is as follows: One of the functions u (say) being known, a first group of $(n-1)^2$ equations determines the mutual ratios of the first derivatives of the $n-1$ other functions $v, w, \cdots$; this first

* Halphen, "Sur un système d'équations différentielles," *Comptes rendus*, vol. 92 (1881), p. 1101.

group of equations is identical with that which we find when reducing two quadratic forms in n variables, the variables being connected by a linear relation, to sums composed of the same squares. The equations thus obtained for each of the functions $v, w, \cdots$, give rise to the conditions of integrability which should evidently contain the derivatives of u up to the third order inclusive. By attempting to construct these conditions in the simplest form we find two different groups of third order equations for u. The first equations, $\frac{1}{2}(n-1)(n-2)$ in number, are analogous to the equation formed in the case of three variables, and are possessed, as was also that equation, of the property of assuming a simple form by the aid of the second derivatives of the function

$$H = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \tag{7}$$

The equations of the second group, $\frac{1}{6}(n-1)(n-2)(n-3)$ in number, appear only when n is greater than 3. In order to interpret these results in geometric language Darboux introduces the generalized notion of a normal to a surface; a generalization which carries with it the extension of the definitions of principal directions and lines of curvature. In space of n dimensions, there are $n-1$ principal directions and consequently $n-1$ systems of lines of curvature for each surface. Two different principal directions are simultaneously orthogonal and conjugate. These notions lead at once to the following generalization of Dupin's theorem: If a surface is part of a completely orthogonal system, the surfaces which belong to other families admit of as normals at each point of the surface considered the principal directions of this surface; consequently, taken $n-2$ by $n-2$, they cut this surface along one of its lines of curvature. It is interesting to note a new property here, namely, that not every surface in a space of n dimensions, n being greater than 3, can be part of a completely orthogonal system; in order that this be possible it is necessary that its lines of curvature be *coördinated*. Darboux says that the lines of curvature are coördinated when we can find $n-1$ functions, of which *one only* varies upon each line of curvature. The second group of equations referred to above give the necessary and sufficient condition that the lines of curvature of the surface whose parameter is u be coördinated. The theorem, that if the lines of curvature of a surface are coördinated the surface can be a member of a completely orthogonal system, is established by generalizing the theory

of parallel surfaces. The remainder of the chapter is occupied with the study of various orthogonal systems, among which are 1° those systems containing a family of the form

$$u = X_1 + X_2 + \cdots + X_n,$$

where u is an arbitrary constant and X_i a function of x_i alone, and 2° those possessing a family defined by the equation

$$u = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

in which the m's are constants; together with the determination of the lines of curvature of different surfaces among which we note the tetraedral symmetric surfaces, called attention to by Lamé* and studied extensively by De la Gournerie.† Any account of this chapter, however brief, would be incomplete without remarking a beautiful generalization of propositions due to Bouquet and Serret,‡ and the fact that the results of the chapter give the lines of curvature of a great number of surfaces of the third order in ordinary space.

This chapter concludes the first book of the work which is divided into two books of equal length and each of the same number of chapters. The first book is devoted to the equation of the third order as we have seen; the second book takes up the study of curvilinear coördinates. Before attempting a brief digest of the rich material of the second division of the volume it may be remarked that Darboux has spared no pains to make his work immediately available to the student. The carefully prepared analytical summaries preceding each chapter and the judicious use of italics throughout the text have greatly increased the usefulness of an otherwise valuable treatise. The reader will find the author's theory of surfaces almost indispensable as an auxiliary because of the numerous references to that classic work.

Book II. Curvilinear Coördinates.

1. It is the object of this chapter, the first one of the second book, to present the formulæ which Lamé has given in his

* Lamé, "Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie," Paris, 1818.

† De la Gournerie, "Recherches sur les surfaces réglées tétraédrales symmétriques." Paris, 1867. This work consists of three memoirs which appeared in the *Recueil des Savants étrangers*, 1865–66.

‡ Serret, J. A., "Mémoire sur les surfaces orthogonales," *Liouville's Journal*, 1st series, vol. 12 (1847), p. 241.

"Leçons sur les coördonnées curvilignes et leurs diverses applications" by applying and extending his method to completely orthogonal systems in n variables; the results have been given in part in Darboux's memoirs of 1866 and 1878 already referred to and cited in footnotes of this review. After defining the elements of an orthogonal linear substitution by the formulæ $X_i^k = H_i \frac{\partial \rho_i}{\partial x_k}$, and deducing the differential relations to which they give place, Darboux introduces the new quantities β_{ik} which verify two different systems of partial differential equations of the first order. In the process of developing the general method Darboux finds the following generalization of a theorem of Combescure:*—To every orthogonal system we can join an infinity of others which depend on n arbitrary functions of one variable. The general method of determining an orthogonal system consists of the following steps: 1° the integration of the equations for the functions β_{ik}; 2° that of the system for the functions H_i; 3° the integration of the system which determines the functions U_i; then 4° the integration of the expressions

$$du = H_1 U_1 d\rho_1 + \cdots + H_n U_n d\rho_n$$

will give the functions $x_1, \cdots, x_n$ and the solution of the problem will be achieved. The author applies this general method to find the solution of a fundamental problem, namely, the most general resolution of the equation

$$dx_1^2 + \cdots + dx_n^2 = \frac{1}{h^2}(d\rho_1^2 + \cdots + d\rho_n^2).$$

The solution is entirely similar to the one long known for the case of three variables and is furnished by a generalized inversion preceded or followed by a displacement. That to a system in elliptical coördinates we can make correspond an unlimited series of algebraic orthogonal systems is one of the remarkable results called to mind in the presentation of the different methods by which orthogonal systems in n or fewer variables can be obtained from an orthogonal system in n variables. Darboux derives a geometrical theorem which generalizes the Gaussian notion of spherical representation and uses it in the investigation of surfaces in an n-dimensional space whose lines of curvature are coördinated.

*Combescure, E., "Sur les déterminants fonctionnels et les coördonnées curvilignes," *Annales de l'École Normale*, 1st series, vol. 4 (1867), p. 93.

The latter problem demands the determination of all completely orthogonal systems in a space of $n - 1$ dimensions, a determination which is effected by methods analogous to those followed in seeking a surface having a given spherical representation. The chapter terminates with a generalization of the theory of cyclic systems and its extension to space of n dimensions.

2. This chapter studies the properties of triply orthogonal systems by the aid of the displacement of a trirectangular triedron (T) formed by the normals to the three coördinate surfaces which intersect in each point of space. Preparatory to making this application of the theory of the mobile triedron Darboux reproduces a body of results already obtained in his Theory of Surfaces, namely, the partial differential equations connecting the rotations and the translations, the determination of a triedron whose rotations and translations are given *a priori*, the linear system whose integration gives the direction cosines of the axes of this triedron, and the general formulæ which determine the displacement of a point defined by its coördinates relative to the moving triedron. As observed above, this general theory is applied to curvilinear coördinates by taking as moving triedron the one whose sides are normal respectively to the three coördinate surfaces; three of the rotations are zero, the others are expressible in terms of the coefficients H, H_1, H_2 of the lineal element, and of their derivatives; in developing the theory of this particular triedron the quantities $\beta_{\alpha\beta}$, introduced in the preceding chapter, play an important part. The formulæ for the projections of a displacement on the axes of the triedon lead to a new demonstration of Dupin's theorem. After deriving the expressions of the radii of curvature of the coördinate surfaces and the law of variation of the six principal curvatures discovered by Lamé, Darboux defines Lamé's differential parameter of the first order ΔU, and derives its expression and that of the form $\Delta\,(U, V)$ in curvilinear coördinates. He then uses the theorem of Stokes and Carl Neumann to find the differential parameter or linear invariant of the second order, and shows in a most interesting manner how the introduction of this invariant may be attached to the study of certain properties of general point transformations. For all transformations of this kind there are in general three lineal elements issuing from a point to which correspond parallel elements; if these three lineal elements are always to form a trirectangular triedron, Darboux finds it necessary and sufficient that the transformation shall be defined by the formulæ

$$X=\frac{\partial U}{\partial x}, \quad Y=\frac{\partial U}{\partial y}, \quad Z=\frac{\partial U}{\partial z},$$

where U is any function of x, y, z. There exists then an equation of the third degree which makes known the dilatations of elements whose directions are unaltered, and the roots of this equation are the invariants. Lamé's differential parameter of the second order is none other than the sum of the roots of this equation. The problem, to determine the transformations of the above nature for which the elements whose direction is not changed in space are normal to three families forming necessarily a triply orthogonal system of curvilinear coördinates, leads to the result that there are an infinite number of transformations of this kind corresponding to a triple system given *a priori*. The determination of the latter transformations depends on the integration of three linear partial differential equations which are satisfied by the same function. The final study of the chapter is a complement of the theory of the displacement of the triedron (T). If we consider the most general system of oblique curvilinear coördinates, the lineal element of space takes the form

$$ds^2 = \Sigma\Sigma A_{ik} d\rho_i d\rho_k;$$

it may be proposed to find all the differential relations which exist among the A_{ik}, differential relations analogous to those which we owe to Lamé for the case of orthogonal coördinates. This question can be solved by taking a triedron (T) occupying a particular position relative to the tangent planes of the coördinate surfaces, but Darboux employs a much more elegant method which rests upon the consideration of the different decompositions of the above quadratic form into squares and which leads to six equations simultaneously necessary and sufficient.

3. This chapter is devoted to the investigation of a particular triple system. Lamé has attached great importance in his Leçons to systems composed of three *isothermal* families. His definition and criterion, verified in the case of the system of confocal ellipsoids, show that there exists at least one triple system simultaneously orthogonal and isothermal. Lamé proposed to determine all systems of this kind. Bertrand* has shown that the surfaces which compose these particular systems are *isothermic*, *i. e.*, they are divided into infinitesimal squares by their lines of curvature; but this

*Bertrand, J., "Mémoire sur les surfaces isothermes orthogonales," *Liouville's Journal*, vol. 9 (1844), p. 117.

property is not characteristic, it belongs, for example, to the system of confocal cyclides, which is not isothermal. Hence by proposing to find all triple systems composed of isothermic systems, if the problem is successfully solved we are assured of obtaining not only the isothermal systems sought by Lamé, but also systems more general. Lamé has shown that if ρ, ρ_1, ρ_2 designate the elliptic coördinates of a point in space, the equation of heat admits of an infinity of solutions of the form $f(\rho) f_1(\rho_1) f_2(\rho_2)$. Darboux shows that all triply orthogonal systems, for which an analogous proposition can be formulated are composed of isothermic surfaces. All these remarks lead to an attack on the more general problem, that of the determination of the triple systems composed of isothermic surfaces. Darboux finds that the equations for determining H, H_1, and H_2 possess three types of solutions. The first type gives three triple systems: 1° a family of parallel planes and two families of isothermal cylinders; 2° a family of concentric spheres and two families of isothermal cones; 3° a family of planes passing through a straight line and two families of surfaces of revolution whose axis is the line and whose meridians form an orthogonal and isothermal system. To these systems should be added those obtained from them by inversion. The second type does not furnish a solution of the problem and should be rejected.

4. The third type of solution is the subject of the fourth chapter. It corresponds to the values

$$H = \frac{(\rho_1 - \rho)^{-h}(\rho - \rho_2)^{-h}}{M \sqrt{a}}, \quad H_1 = \frac{(\rho_2 - \rho_1)^{-h}(\rho_1 - \rho)^{-h}}{M \sqrt{a_1}},$$

$$H_2 = \frac{(\rho - \rho_2)^{-h}(\rho_2 - \rho_1)^{-h}}{M \sqrt{a_2}}.$$

In these formulæ M is any function, h is a constant and a_i is a function only of ρ_i. By expressing that certain equations of condition given in the preceding chapter are verified, Darboux obtains the following solutions: 1° $h = -\frac{1}{2}$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ are identical polynomials of the fifth degree; the corresponding triple system is that formed by the confocal cyclides and their varieties. 2° $h = \frac{1}{2}$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ are identical polynomials of the third degree; the corresponding system is transcendental.* 3° $h = 1$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ are polynomials of the second degree whose

* For the study of the hypotheses, $h = \frac{1}{2}$ and $h = 2$, the author refers the reader to the memoirs of his already cited.

sum is zero; the corresponding system is composed exclusively of those surfaces to which Darboux gave the name cyclides of Dupin. The author demonstrates that this system is identical with those studied in the third chapter of the first book. 4° $h = 2$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ reduce to constants whose sum is zero; the corresponding system is imaginary.* In the course of the discussion Darboux shows how the value of M corresponding to each solution may be determined.

5. Among the systems determined in the two preceding chapters there are two classes which are to be the objects of study in this chapter: 1° those composed of three isothermal families; 2° those for which the equation of heat admits of an infinity of solutions of the form

$$P(\rho, \rho_1, \rho_2)\, f(\rho) f_1(\rho_1) f_2(\rho_2),$$

P being a determinate function while the f's may be chosen in an infinite number of ways. It is proposed to determine these two particular classes. The author commences by seeking to determine the isothermal systems, considering successively the different solutions obtained in the two preceding chapters. The isothermal systems composed of a family of parallel planes and two families of isothermal cylinders, or of a family of concentric spheres and two families of isothermal cones, or a family of planes passing through an axis and two families of confocal quadrics of revolution, are first found, then the system of confocal quadrics and finally an imaginary system. This last isothermal system was pointed out by Combescure† but not determined by him; it plays no rôle in mathematical physics, but has a geometric interest; Darboux finds that it is formed by cyclides of Dupin which are imaginary, and of the third degree. The author follows an analogous method for the determination of the second class of triple systems noted above. By the use of a fundamental lemma relative to inversion, due to Lord Kelvin,‡ Darboux proves that the systems sought reduce to the single system of confocal cyclides and their varieties.§ He also establishes this prop-

*See footnote on the preceding page of this review.

† *Vide loc. cit.*

‡ Thomson, W., "Extrait de deux lettres adressées à M. Liouville;" Liouville: "Note au sujet de l'article précédent," *Liouville's Journal*, 1st series, vol. 12 (1847), pp. 256 and 265.

§ This property was established by Wangerin for cyclides having principal planes; Darboux demonstrated the result directly and extended it to cyclides of the most general form. See Wangerin, A., "Ueber ein

erty of confocal cyclides by a direct demonstration in which he employs pentaspherical coördinates and a remarkable form which the equation of heat assumes in this system of coördinates. The chapter ends with a remark pointing to an extension of some of the above results to systems of curvilinear coördinates the most general.

6. The sixth chapter of the second book and last chapter of the volume is devoted to those triple systems which Darboux has named the triple systems of Bianchi. These are the triple systems which contain a family of surfaces of constant total curvature, the curvature varying from surface to surface of the family according to any law. This application of the general method made by Bianchi has its origin in a theorem of Weingarten which foreshadows the existence of families of Lamé composed of surfaces of constant and equal total curvature; Weingarten's theorem is as follows: Given a surface (S) having the constant total curvature $1/k$, if we take upon the normal at each of its points an infinitesimal length MM' proportional to $\cos d/\sqrt{k}$, d designating the geodesic distance of M from a fixed point A of (S), the surface (S') described by the point M' is also of constant curvature $1/k$ and together with (S) is part of a family of Lamé. This theorem makes possible the construction one after another of surfaces having the same constant total curvature which depend upon four arbitrary functions and form a family of Lamé. After solving a problem in the theory of deformation of surfaces naturally suggested by Weingarten's theorem and remarking properties of certain cyclic systems attached to a surface applicable on a surface of revolution, Darboux presents an exposition of the researches of Bianchi. By discarding the special case in which the surfaces sought are surfaces of revolution, we obtain an elegant form of the lineal element; this form depends on a single function ω which must satisfy three partial differential equations of the second and third order. The study of this system of partial differential equations reveals the fact that its general integral depends upon five arbitrary functions of one variable. Among the general properties of the triple systems to which we are led we note that the transformation of Bäcklund can be applied to them as Bianchi has shown. After examining the systems of

dreifach orthogonales Flächensystem, gebildet aus gewissen Flächen vierter Ordnung," *Crelle's Journal*, vol. 82 (1876), p. 145; Darboux, "Sur l'application de méthodes de la physique mathématique à l'étude des corps terminés par des cyclides," *Comptes rendus*, vol. 83 (1876), pp. 1037, 1099.

Weingarten in particular, Darboux concludes the volume by determining all triple systems for which the nine quantities H, β_α depend on a single variable α; the result is a system composed exclusively of helicoids having constant total curvature.

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PRINCETON, NEW JERSEY.

THE NEW MATHEMATICAL ENCYCLOPAEDIA.

IT is an established belief that mathematics like the classics stopped growing long ago. The chemist, biologist, physicist, and other scientists look with complacency at the gigantic strides their branches of learning have taken in the last generation and rather pity their colleagues, the mathematicians. According to them, the golden age of our science dates back two thousand years ago when Euclid, Archimedes, Apollonius and Diophantes flourished. It is true that some time afterwards algebra and trigonometry, analytical geometry and the calculus were invented; they have, however, been long since perfected and mathematicians spend their time teaching this ancient body of facts, improving here and there a small detail and solving ingenious problems which they devise to test each other's skill. How different the real state of affairs is! No science presents a more intensely active and vigorous condition than ours. Indeed the growth of mathematics has been so rapid in the last century, the discoveries and inventions so numerous and their importance so far reaching that it is permitted only to a very few extraordinary minds still to overlook the whole field of mathematics as it stands to-day. One has only to recall a few of the great theories which have sprung up in these later days and the numberless special theories and ramifications they have given rise to. First of all that great leviathan, the function theory, embracing as special topics of limitless extent the theory of elliptic, hyperelliptic, abelian, modular, and automorphic functions not to mention others. Of still younger date is the theory of groups finite and infinite, with their far-reaching applications. We have then the modern theory of invariants, the new theory of algebraic numbers, the non-Euclidian geometry, the theory of algebraic transformations and correspondences, Cantor's theory of multiplicities, the partial

differential equations of the potential theory, the modern theory of linear differential equations inaugurated by Riemann and Fuchs, the arithmetical theories of Kronecker, etc., etc.

But there is no need to enumerate further. The field of mathematical activity is so boundless, the details so rich and varied that the ordinary mind falls into a state of helpless despair when occasionally contemplating it as a whole. The days when, with industry and patient perseverance, one could embrace all that was known in mathematics or even all that is fundamental and important have long since passed by. But a more hopeless state of affairs has begun to prevail; we not only cannot embrace the whole but the most of us cannot even keep up with the latest advances in a few special theories. We find ourselves compelled more and more to limit the range of our vision, to confine ourselves even to a part of a single theory. Nor is this wonderful when we reflect how rapidly mathematical literature is increasing. In the course of a single year 2,000 books and memoirs on mathematical subjects appear, while the number which have appeared during the last generation is estimated at about 50,000.

Specialism is thus the motto of the day, it is not a matter of choice but of necessity. But against this state of bondage two forces are constantly at work. In the first place it is an inherent, innate principle of the human mind to have a lively interest in what immediately surrounds it, and this is true in the world of mathematics as everywhere else. The normal mathematician cannot help longing to know something of the regions which border on his own special field of study. He is as eager to travel through some of these unknown countries, admiring their peculiar beauties and acquiring fresh insight into the deeper relations of things, as any tourist for a jaunt through India or Zululand.

This is the ideal force; the other force is of very practical and urgent nature. Every mathematician knows that the objects of the mathematical world hang together more or less intimately. No theory stands isolated from all others. The systematic attempt of purists to banish all imagined foreign elements from their theories must lead to utter sterility and self destruction. No mathematician can afford to be altogether a specialist. Indeed are we not all of us from time to time forced to leave our chosen field of research and pick up as rapidly as possible the essentials of some widely different theory in order to overcome a difficulty in our path, or to read an important paper engendered

by a different circle of ideas than our own? Often we find it impossible to do this. The facts we are in need of are of so special a nature, they lie so embedded in a mass of complicated concepts and theories that the labor of unraveling them from their surroundings is too great to be undertaken, at least for the present. That this is a grave misfortune is manifest; that it must occur to each individual mathematician more and more frequently as our science becomes not only broader but also more specialized is equally patent. In this dilemma mathematicians have sought assistance in several directions. In the first place attempts have been made to make existent knowledge more accessible by having exhaustive and up to date bibliographies prepared. I mention here the "Répertoire bibliographique des sciences mathématiques" appearing under the auspices of the Société Mathématique de France. In the second place the Deutsche Mathematiker-Vereinigung, with characteristic energy, has instituted a series of admirably prepared reports on the various departments of mathematics. Already reports on invariants, algebraic functions and their integrals, algebraic numbers, and synthetic geometry have appeared and many others are in preparation. These reports are historical.* Going back to the first germs, the development of the theory in all its principal branches is traced down to the present day. They presuppose upon the part of the reader more or less familiarity with the modern state of the particular theory in question; their object is to give him a more comprehensive and deeper knowledge of it by showing him how the various branches of the theory arose, how each was related to and influenced by its surroundings, and how it in turn reacted on them. Hardly a science exists where it is so important to have a historical knowledge of the subject as in mathematics.

Both these undertakings fill a long felt want. But there exists still another need more pressing, more constantly felt than either of these. It is the need of a mathematical encyclopedia. Attempts to meet this crying demand have already been made. We have in German Hoffmann and Natani's Mathematisches Wörterbuch in seven volumes, (Berlin, 1858–67) Schlömilch's Handbuch der Mathematik in two volumes, (Breslau, 1879–81). In English we have Carr's Synopsis of Mathematics in one volume, (London, 1886). None of these are of much service to the professional mathematician of today. Finally we have a work

* Hilbert's report is an exception.

of much more advanced character and consequently of greater usefulness in J. G. Hagen's Synopsis der höheren Mathematik, of which two volumes in large quarto have appeared (Berlin, 1891–94). A third and final volume will appear shortly. To give an account of mathematical theories as they stand today unfinished and scattered in a thousand memoirs is of course utterly beyond the powers of a single man. Father Hagen has accordingly limited his undertaking to a digest of the principal treatises, and to bibliographical references confined for the most part to those given in these works.

But to fully satisfy the demands of the mathematician of today an undertaking on a far larger scale is necessary. A synopsis based on treatises must necessarily be years behind the present state of the science; on the other hand to prepare a work which shall give the latest developments of our science is not the labor of one man but of scores. To have each subject adequately treated requires that each contributor should have made his subject one of special study and research; he must be familiar with all its phases, its multitudinous ramifications, its history and literature. Such men are difficult to acquire. They are already overburdened partly by their own original investigations, partly by their university duties. To write one of these articles would be a great sacrifice. In the case of our great popular literary encyclopaedias like the Encyclopaedia Britannica or Brockhaus's Conversationslexikon the sales are so large that the contributors receive a comfortable compensation for their labor; but this cannot be calculated upon for a mathematical encyclopaedia. Pecuniary compensation must be nominal. Besides the contributors there must be an efficient board of editors. Their labors are still more arduous, the sacrifice of their time proportionately greater. Still another important factor must be considered. Since the sales of such a highly technical work must necessarily be relatively limited, what publisher would be willing to assume a risk whose financial prospects were so problematical? Under these unpropitious circumstances it might well seem that a mathematical encyclopaedia as here sketched would never be more than a dream.

That to-day, however, the Encyclopaedia is indeed a fact, that already a first Heft has appeared and others are to follow in rapid succession is largely due to the genius, energy and courage of a single man, Felix Klein. Who besides him could set springs in action which would bring together in enthusiastic coöperation two score or more of specialists

from all countries to contribute articles, find efficient editors to unify and make ready for the press the mass of material sent in, and finally by winning the support of the academies of Göttingen, Munich, and Vienna guarantee to a certain extent the financial success of the undertaking to the publishers?

Having shown how great the need of such an encyclopaedia is, and indicated some of the difficulties to be overcome in its realization we wish now to give a few details which will enable the readers of this BULLETIN to form a more accurate idea of this great work. According to what scheme is the work planned, who are the men entrusted with its execution and what are the names of some of the principal contributors?

In the preface to the first number we read: "The object of the Encyclopaedia is to give in a concise form, adapted to rapid orientation, but with all possible completeness, an account of mathematical sciences as they exist today, together with carefully prepared bibliographical notices which sketch their developments since the beginning of the present century. Not only pure mathematics, but also its applications to physics, astronomy, geodesy, as well as the various technical branches will be treated, the idea here being on the one hand to show the mathematician what problems the applications give rise to, on the other to inform the astronomer, the physicist, the engineer, what answer mathematics gives to the same." Demonstrations will not be given, at most only sketched; only a limited knowledge of a given subject will be presupposed. The work is to appear in six volumes. Vol. I will embrace arithmetic, algebra, theory of numbers, probabilities and the adjustment of observations, finite differences, numerical calculations. Vol. II is devoted to the theory of functions of real and complex variables, including such subjects as the differential and integral calculus, differential equations, continuous groups, calculus of variations, general function theory, spherical harmonics, Abelian and automorphic functions, etc. Vol. III deals with geometry; topics are pure geometry, the foundations of the applications of algebra and analysis to geometry, algebraic geometry, differential geometry. Vols. IV and V are to embrace the applications of mathematics, while Vol. VI will be devoted to historical, philosophical and pedagogical questions. It will also contain a comprehensive index of the whole work. So much for the plan of the work. Regarding its execution we must be brief. The editors of the Encyclopaedia are H. Burkhardt and W.

F. Meyer; assisting them is a commission appointed by the academies already mentioned consisting of F. Klein, G. v. Escherich, H. Weber, L. Boltzmann, and W. Dyck. To give an idea of the kind of men who are contributing articles we mention *quite at random* the names: Bachmann, Burkhardt, Castelnuovo, Dyck, Enriques, Fricke, Hilbert, Hölder, F. Klein, Kneser, Krazer, v. Mangoldt, Maurer, W. F. Meyer, Netto, Painlevé, Pincherle, Pringsheim, Runge, G. Scheffers, Schönflies, Segre, Sommerfeld, Staeckel, Study, H. Weber, E. v. Weber, Wirtinger and Zeuthen. To this list of names Americans will be gratified to see added those of M. Bôcher, J. Harkness and W. F. Osgood.

We close wishing the Encyclopaedia all success. May it become a familiar handbook to every American mathematician. In our opinion no mathematical work has ever appeared which has such unquestionable claims to be found in the library of every mathematician. No one can make constant use of its wealth of information without deriving invaluable assistance. We have no doubt it will enjoy the most widespread popularity not only with mathematicians but with physicists, astronomers, and engineers in every land. May Americans be quick to avail themselves of this veritable vade-mecum.

JAMES PIERPONT.

YALE UNIVERSITY,
December, 1898.

ERRATA.

VARIOUS errata occurring in previous numbers of the present volume of the BULLETIN are herewith noted:

On page 2, line 10, the name of Professor W. E. Story should be inserted in the list of members present at the Summer Meeting.

On page 48, line 19, *for* Sadlerian *read* Sedleian.

On page 100, line 33, *for* not known *read* now known.

On page 156, lines 31–35, a regrettable error occurs, arising from mistaken identity. Mr. Horace T. Eddy was appointed instructor in electrical engineering and physics at Union College at the beginning of the present academic year. Professor Henry T. Eddy continues to fill the chair of engineering and mechanics in the University of Minnesota.

NOTES.

At the meeting of the Cambridge Philosophical Society held on October 31, 1898, papers were presented by Mr. H. F. Baker, "On Mittag-Leffler's theorem;" by Mr. A. Berry, "On the evaluation of a certain determinant which occurs in the theories of statistics and of elliptic space;" by Mr. J. H. Grace, "On metrical relations between linear complexes;" by Mr. A. E. Western, "On certain systems of quadratic complex numbers."

The Mathematical Society of Edinburgh held its annual meeting on November 11, 1898, and elected the following officers for the current session: President, Dr. Alexander Morgan; Vice-president, Mr. R. F. Muirhead; Honorary secretary, Mr. J. W. Butters; Honorary treasurer, Mr. F. Spence; Editors, Mr. John Dougall, Mr. Charles Tweedie, Dr. C. G. Knott; Committee, Messrs. J. D. H. Dickson, George Duthie, A. Lindsay. At this meeting papers on "Systems of circles analogous to Tucker's circles," by Mr. J. A. Third, and on "The geometrical theory of the hyperbolic functions," by Mr. W. L. Thomson, were read. On the motion of Professor G. A. Gibson, a committee was appointed to consider the treatment of proportion in elementary mathematics.

The Trustees of the British Museum have recently issued a fac-simile, consisting of twenty-one plates, of the celebrated Rhind mathematical papyrus, prefaced by an introduction by Dr. E. A. Wallis Budge, keeper of the Egyptian and Assyrian antiquities. This papyrus has been the object of considerable discussion on the part of mathematicians and Egyptologists ever since the late Dr. Samuel Birch published an account of its contents in the *Zeitschrift für Ægyptische Sprache* in 1868, and a large body of students in each field will be grateful for the possession of the actual text. The text is written throughout in hieratic, but its actual date is not quite certain. Dr. Budge assigns it to a period not earlier than the beginning of the eighteenth dynasty, about 1700 B. C., but adds that the text goes back to a more remote period. It was probably a copy of a papyrus written in the Hyksos period, about 2000 B. C., by a scribe Aäh-mes, who stated that he himself copied an original work of the time of Amen-em-hāt III., a king of the twelfth dynasty, about 2300 B. C. The number of *Nature* for November 24, 1898, contains a sketch of the contents of the papyrus.

THE press of B. G. Teubner announces that a work entitled "Kurzgefasste Vorlesungen über verschiedene Gebiete der höheren Mathematik" is in preparation by Dr. ROBERT FRICKE. The work is to consist of two parts, devoted to algebra and geometry, and analysis and theory of functions, respectively, and will be issued in octavo form bound. It is to assume on the part of the reader familiarity with the elements of descriptive geometry, analytical geometry, and the infinitesimal calculus; its object is pedagogic and it will attach chief importance to the applications of the science. Designed primarily for students of physics, astronomy, and engineering, and for those students of pure mathematics who have no desire to specialize, the work is to be constructed on lines similar to those followed in Merriman and Woodward's "Higher mathematics, a text-book for classical and engineering colleges."

ONE of the earliest works to bear the imprint of 1899 is Professor E. CESÁRO'S "Elementi di calcolo infinitesimale, con numerose applicazioni geometriche," from the press of Lorenzo Alvano, Naples. This volume is a fitting sequel to the author's "Corso di analise algebrica" and a suitable review of it will appear in the BULLETIN.

THE last catalogue, Number 17, December, 1898, of the Fratelli Bocca, of Rome, contains over fifteen hundred numbers of works in mathematics, physics and astronomy. A. Hermann, of Paris, has recently issued a new catalogue of his own publications and also Catalogue Number 60 of memoirs and treatises in mathematics and astronomy. The contents of the latter catalogue are drawn largely from the library of the late Professor SOUILLART and comprise more than three thousand numbers in mathematics, over a thousand numbers in astronomy, and about one hundred and fifty portraits of savants. Macmillan and Bowes, of Cambridge, announce a number of standard English mathematical books at greatly reduced prices in a recent circular. Catalogue Number 89, dated 1899, of Oscar Schack, of Leipsic, offers over a thousand numbers in the exact sciences.

CAMBRIDGE UNIVERSITY. The following lectures on mathematical subjects will be delivered during the academic year 1898–99: *Michaelmas term*:—By Professor G. G. STOKES: Hydrodynamics.—By Professor A. R. FORSYTH: Theory of differential equations; Fourier's and other expansion theorems. By Professor G. H. DARWIN: Orbits and perturbations of planets.—By Dr. E. W.

HOBSON: Sound and vibrations.—By Mr. J. LARMOR: Electrostatics and magnetism.—By Mr. A. E. H. LOVE: General theory of wave motion and optical theories.—By Mr. H. F. BAKER: Theory of functions.—By Mr. A. BERRY: Elliptic functions.—By Mr. H. W. RICHMOND: Plane analytical geometry.—By Mr. G. T. WALKER: The electromagnetic field. *Lent term*:—By Professor G. G. STOKES: Physical optics.—By Professor A. R. FORSYTH: Differential equations (continued).—By Professor R. S. BALL: Planetary theory.—By Professor J. J. THOMSON: Electricity and magnetism, electromagnetic waves.—By Dr. E. W. HOBSON: Conduction of heat.—By Mr. J. LARMOR: Electrodynamics and electro-optics.—By Mr. A. E. H. LOVE: Elasticity.—By Mr. H. F. BAKER: Theory of functions (continued).—By Dr. J. W. L. GLAISHER: Elliptic functions.—By Mr. R. A. HERMAN: Hydrodynamics.—By Mr. A. N. WHITEHEAD: Non-Euclidean geometry.—By Mr. G. T. WALKER: Physical optics.—By Mr. G. T. BENNETT: Linear and quadratic complexes. *Easter term*:—By Professor R. S. BALL: Movements of comets.—By Professor J. J. THOMSON: Electricity and magnetism.—By Mr. H. F. BAKER: Solid geometry.—By Dr. J. W. L. GLAISHER: Elliptic functions (continued). *Long vacation*:—By Professor G. H. DARWIN: Potential and attractions; Figure of the earth and precession.

The Examiners for Part II. of the Mathematical Tripos in 1899 are Dr. E. W. HOBSON, Mr. A. N. WHITEHEAD, Mr. W. BURNSIDE, and Mr. G. T. WALKER.

PROFESSOR G. G. STOKES will have completed the fiftieth year of his tenure of the Lucasian Professorship of Mathematics at Cambridge University next summer. A plan is on foot to celebrate the event in a suitable manner on the 1st and 2d of June, 1899. Professor Stokes is the thirteenth mathematician to hold this professorship founded by Henry Lucas in 1663. Isaac Barrow was the first incumbent in 1664, Isaac Newton assumed the chair in 1669, W. Whiston in 1702, N. Sanderson in 1711, J. Colson in 1739, E. Waring in 1760, Isaac Milner in 1798, R. Woodhouse in 1820, R. Turton in 1822, G. B. Airy in 1826, Charles Babbage in 1828, Joshua King in 1839, and G. G. Stokes in 1849. It is interesting to remark further that, excepting Charles Babbage, who held the professorship for eleven years but delivered no lectures, all occupants of the chair elected since 1747–48, the session in which the mathematical triposes were instituted, have been senior wranglers; and

all, with the same exception, elected since 1769, the date of the founding of Dr. Smith's Mathematical Prizes, have been first Smith's prizemen: Waring in 1757, Milner in 1774, Woodhouse in 1795, Turton in 1805 (ranked equal with Christie, second wrangler, as Smith's prizeman), King in 1819, Airy in 1823, STOKES in 1841.

PROFESSORS WILLIAM BEEBE and JAMES PIERPONT have been promoted from assistant to full professorships in mathematics at Yale University.

PROFESSOR F. H. LOUD, of the mathematical department of Colorado College, has been granted leave of absence for one year.

MR. WILLIAM FOX has been appointed assistant professor of applied mathematics in the City College of New York.

MR. R. C. MACLAURIN has been elected to a fellowship in mathematics at St. John's College, Cambridge University.

M. A. MICHEL LÉVY, of Paris, has been elected a corresponding member of the Berlin Academy of Sciences.

THE Munich Academy of Sciences has elected the following mathematicians to membership: Professor L. FUCHS, of Berlin, Professor SOPHUS LIE, of Christiania, and Professor A. PRINGSHEIM, of Munich.

THE Sylvester Memorial Committee (see BULLETIN, 2d Series, vol. 4, p. 239) met recently in London and closed its accounts, the entire amount collected being about eight hundred and seventy-five pounds sterling. It was decided to establish a medal in bronze, to be given every three years together with the interest on the fund for that period, for distinguished work in mathematics.

PROFESSOR CHARLES-MICHEL BRISSE died at Paris, October 13th, 1898, at the age of fifty-five years. The November number of the *Journal de Physique*, with which he was actively connected from the date of its foundation in 1872, contains an appreciative account of his services to science. He was the author of memoirs on the theory of surfaces and on the displacement of figures, of a number of classic papers on actuarial subjects, and of French translations of English and German works on higher mathematics. He published two editions of a course in mathematical physics, the first in collaboration with M. C. ANDRÉ, and the second with M. REVIÈRE. For twenty-four years he was professor of mathematics at the Lycée Condorcet, and also held the posts of répétiteur at the École Polytechnique, supplementary pro-

fessor at the Conservatoire des Arts et Métiers, professor at the École Centrale, and professor at the École des Beaux-Arts.

THE deaths are announced of Sir GEORGE BADEN-POWELL, at the age of fifty-one years, a patron of astronomical science, son of the Rev. Prof. Baden-Powell, the Oxford geometrician and geologist; of Sir JOHN FOWLER, at the age of eighty-one, the celebrated English engineer, designer of the Firth of Forth bridge; and of M. J. N. RAFFARD, at the age of seventy-four, one of the editors of the *Revue de Mécanique*, and distinguished for his contributions to applied mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABHANDLUNGEN zur Geschichte der Mathematik. Heft VIII. Leipzig, Teubner, 1898. 8vo. M. 8.00

BOEGEHOLD (H.). Historisch-kritische Darstellung der Konstruktionen der Fläche zweiter Ordnung aus 9 Punkten. [Diss.] Jena, 1898 8vo. 52 pp.

BIANCHI (L.). Vorlesungen über Differentialgeometrie. Uebersetzt von M. Lukat. (In drei Lieferungen.) Zweite Lieferung. Leipzig, Teubner, 1898. 8vo. pp. 337 to 528. M. 6.60

BOCHOW (K.). Die Formeln für die Summe der natürlichen Zahlen und ihrer ersten Potenzen, abgeleitet an Figuren. Magdeburg, 1898. 8vo. 45 pp. M. 1.00

BOHLMANN (G.). See GENOCCHI (A.).

BUDISAVLJEVIC (E. v.) und MIKUTA (A.). Leitfaden für den Unterricht in der höheren Mathematik. Band II. Grundzüge der Differential- und Integralrechnung. Vienna, 1898. 8vo. 8 and 607 pp. Cloth. M. 10.00

COMSTOCK (E. H.). See SLICHTER (C. S.).

DIEM (G.). Ueber Ellipsen auf einem Ellipsoid, deren Axen gegebenen einfachen Bedingungen genügen, insbesondere über kongruente Ellipsen. [Diss.] München, 1898. 8vo. 42 pp.

FRENZEL (R.). Flächen zweiter Ordnung, die durch die Rotation eines ebenen Gebildes um einen im Raume befindlichen Strahl entstehen. [Progr.] Jägerndorf, 1898. 8vo. 16 pp.

FÜRLE (H.). Ueber die Verwendung des Faberschen Rechenstabes zur Lösung quadratischer, kubischer und biquadratischer Gleichungen. Theil I. Berlin, 1898. 4to. 7 pp. M. 1.00

GEER (P. v.). Leerboek der analytische Meetkunde. Deel I.: Meetkunde van het platte vlak en van de vlakken en rechte lijnen in de ruimte. Leyden, 1898. 8vo. 12 and 266 pp. M. 5.80

GENOCCHI (A.). Differentialrechnung und Grundzüge der Integralrechnung, herausgegeben von G. Peano. Uebersetzt von G. Bohlmann und A. Schepp, mit einem Vorwort von A. Mayer. (In zwei Lieferungen.) Erste Lieferung. Leipzig, Teubner, 1898. 8vo. 224 pp. M. 6.00

GIRIODI (V.). Applicazioni delle principali teoriche algebriche. Torino, 1898. 8vo. 12 pp. Fr. 2.00

HEYMANN (W.). Ueber hypergeometrische Functionen, deren letztes Element speciell ist, nebst einer Anwendung auf Algebra. Chemnitz, 1898. 4to. 44 pp. M. 1.80

JACOB (J.). Zur Einführung in die analytische Geometrie. [Progr.] Leitmeritz, 1898. 8vo. 12 pp.

JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann, herausgegeben von E. Lampe. Vol. 27. Jahrgang 1896. (In 3 Heften.) Heft 2. Berlin, Reimer, 1898. 8vo. pp. 369 to 576. M. 7.00

JOCHNIK (W.). Det vigtigaste af algebraiska Analysen. Stockholm, 1898. 8vo. 93 pp. M. 3.50

JUNKER (F.). Höhere Analysis. Theil I. Differentialrechnung. (Sammlung Göschen, No. 87.) Leipzig, Göschen, 1898. 12mo. 192 pp. M. 0.80

KÖTTER (E.). Die Entwicklung der synthetischen Geometrie. Bericht erstattet der Deutschen Mathematiker-Vereinigung. Leiferung 1. Leipzig, Teubner, 1898. 8vo. 128 pp. M. 4.40

LA COUR (P.). Historisk Mathematik. 2te gennemsete udgave. (In 10 Heften.) Heft 2 to 5. Kopenhagen, 1898. pp. 97 to 160. Jedes Heft M. 0.80

LAMPE (E.). See JAHRBUCH.

LAUNER (A.). Entwicklung der Grundbegriffe des geometrischen Calculs von G. Peano. Autorisierte deutsche Uebersetzung. [Progr.] Salzburg, 1898. 8vo. 24 pp.

LÉVY (L.). Précis élémentaire de la théorie des fonctions elliptiques. Paris, Gauthier-Villars, 1898.

LUKAT (M.). See BIANCHI (L.).

MAYER (A.). See GENOCCHI (A.).

MIKUTA (A.). See BUDISAVLJEVIC (E. v.).

MOHRMANN. Bestimmung der Koefficienten in der Potenzreihe die durch Umkehrung einer gegebenen Potenzreihe entsteht, mit Anwendungen. II. [Progr.] Arnstadt, 1898. 4to. 18 pp.

MONTESANO (D.). Una estensione del problema della proiettivita a gruppi di complessi e di congruenze lineari di rette. Milano, 1898. 4to. 45 pp.

NETTO (E.). Vorlesungen über Algebra. (In 2 Bänden.) Band II., erste Lieferung. Leipzig, Teubner, 1898. 8vo. 192 pp. M. 6.00

OLESON (R.). Geometrisk Begynderbok. Uddrag af La Cour, Historisk Matematik, med en del aendringer og tilföjelser. Vejle, 1898. 8vo. 56 pp. M. 1.80

PEANO (G.). See LAUNER (A.).

RICHTER (A.). Beitrag zur Lösung des Apollonischen Berührungsproblems. Wandsbeck, 1898. 4to. 30 pp. M. 1.80

RIEDEL (E.). See SCHURIG (R.).

SCHEPP (A.). See GENOCCHI (A.).

SCHURIG (R.) und RIEDEL (E.). Katechismus der Stereometrie, mit einem Anhang über Kegelschnitte, sowie über Maxima und Minima. Leipzig, 1898. 8vo. 10 and 278 pp. Cloth. M. 3.50

SIMON (M.). Analytische Geometrie des Raumes. (Sammlung Göschen, No. 89.) Leipzig, Göschen, 1898. 12mo. 200 pp. Cloth. M. 0.80

SOBOTKA (J.). Beitrag zur infinitesimalen Geometrie der Integralcurven. Vienna, 1898. 8vo. 55 pp. M. 1.20

SLICHTER (C. S.) and COMSTOCK (E. H.). Harmonic curves of three frequencies and the real singularities of harmonic curves of three frequencies. (*Transactions of the Wisconsin Academy of Sciences.*) Madison, 1898. 8vo. 16 pp.

TENGLER (F.). Construction der conjugierten Durchmesser und Achsen eines Kegelschnittes, der einem gegebenen polar reciprok ist. [Progr.] Klagenfurt, 1898. 8vo. 12 pp.

VAZQUEZ (M.). La solucion algebraica del grado *n* que dicen que Abel demostro imposible, y asi lo afirmaban, después de Abel, todos los Algebristas. Oviedo, 1898. 8vo. 57 pp. Fr. 5.00

ZÜGE (H.). Allgemeine Regeln über die Kennzeichen der Teilbarkeit dekadischer Zahlen. [Progr.] Wilhelmshaven, 1898. 8vo. 28 pp.

II. ELEMENTARY MATHEMATICS.

BASELER (G.). See MEHLER (F. G.).

BLAINE (R. G.). Quick and easy methods of calculating; simple explanations of theory and use of slide-rule, logarithms, etc., with examples. London, Spon, 1898. 12mo. 156 pp. 2s. 6d.

BRIGGS (W.) and BRYAN (G. H.). A middle algebra; based on the algebra of Radhakrishnan. (Univ. tut. series.) London, Clive, 1898. 8vo. 362 pp. 3s. 6d.

——The tutorial Algebra. Part II.: Advanced course, based on the algebra of Radhakrishnan. (Univ. tut. series.) London, Clive, 1898. 8vo. 604 pp. 6s. 6d.

BROWNE (F. J.). See USILL (G. W.).

BRYAN (G. H.). See BRIGGS (W.).

CHAUMEIL (J.) et MOREAU (G.). Troisième livre d'arithmétique. (Complément des deux premiers livres. Six cents exercices et problèmes.) Partie de l'élève. 2e édition. Paris, Larousse, 1898. 12mo. 334 pp. Fr. 1.50

COMTE (F.) et MINGAM (J.). L'arithmétique à l'école primaire (système métrique, géométrie usuelle et comptabilité.) Cours élémentaire. Nouvelle édition, contenant onze cents exercices d'application et problèmes. Paris, Delarue, 1898. 16mo. 8 and 151 pp.

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